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On the quantum superalgebra $U_q(gl(m,n))$ and its representations at roots of 1

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1 Introduction

Let p be a prime, and let η be a pth root of unity in the complex field $\mathbb{C}$. Let $\mathfrak{g}$ be the Lie algebra of the semisimple and simply connected algebraic group G defined over $\overline{\mathbb{F}}_p$. Lusztig defined in [14] a remarkable finite dimensional Hopf algebra arising from quantum group $U_\eta(\mathfrak{g})$. Using which he conjectured that the representation theory of G and that of $U_\eta(\mathfrak{g})$ are the same under certain range of weights. The conjecture is now known to hold for p large than certain number(see [1, 6]).

The main purpose of the article is to extend the Lusztig's conjecture to the super case when $\mathfrak{g}$ is the general linear Lie superalgebra $gl(m,n)$ and G is the linear algebraic supergroup $\mathrm{GL}(m,n)$. Defined in [3, 12], G is the functor from the category of commutative superalgebras to the category of groups defined on a commutative superalgebra A by letting $G(A)$ be the group of all invertible $(m+n)\times(m+n)$ matrices of the form

$$g = \begin{pmatrix} W, & X \\ Y, & Z \end{pmatrix}$$

where W is an $m\times m$ matrix with entries in $A_{\bar{0}}$, X is an $m\times n$ matrix with entries in $A_{\bar{1}}$, Y is an $n\times m$ with entries in $A_{\bar{1}}$, and Z is an $n\times n$ matrix with entries in $A_{\bar{0}}$. The relation between the category of G-modules and category of $\mathfrak{g}$-modules given in [3, 12] is similar to the one between the category of modules for algebraic groups and the category of modules for their Lie algebras [9].

The quantum deformations of the Lie superalgebra $\mathfrak{g} = gl(m,n)$ are given in papers such as [2, 20, 21]. But they are defined all differently. The one given in [2] or [20] has one additional generator. Besides, the proof of the PBW theorem in [20] involves topological methods. We adopt here the definition given in [21]. It should

be noted that the proof of the PBW theorem in [21, Prop.1] is incorrect, where the presumed basis is not shown to be linearly independent.

In this paper we first give a purely algebraic proof of the PBW theorem using the methods given by Jantzen [8] and Lusztig [17]. Using the PBW theorem, we then give a description of $U_q(gl(m,n))$ in terms of generators and relations following Lusztig [14]. With these results, we propose a conjecture for the super case, and prove that the conjecture follows from the Lusztig's conjecture provided that the highest weight is *p-typical* (see Sec. 2.1 for definition). Then we establish the Lusztig's tensor product theorem for the quantum supergroup $U_q(gl(m,n))$. It turns out that this theorem bears much resemblances to the one for the general linear algebraic supergroup given in [12].

The paper is arranged as follows. In Sec. 2 we define the quantum superalgebra $U_q(gl(m,n))$ and study the braid group action on it. In Sec. 3 we study the superalgebra structure of $U_q(gl(m,n))$. In Sec. 4, we give an algebraic proof of the PBW theorem for $U_q(gl(m,n))$. In Sec. 5, we give a description of $U_q(gl(m,n))$ in terms of generators and relations. In Sec.6, we extend the Lusztig's conjecture to the case $U_q(gl(m,n))$ and prove that the conjecture follows from the Lusztig's conjecture in case of a p-typical weight. Finally, we prove in Sec. 7 the Lusztig's tensor product theorem for $U_q(gl(m,n))$.

2 Preliminaries

2.1 The quantum deformation of $gl(m,n)$

Let $\mathfrak{g} = gl(m,n)$ be the general linear Lie superalgebra, and let $\overline{U}(\mathfrak{g})$ be its universal enveloping superalgebra.

Set

$$\mathcal{I}_0 = \{(i,j)|1 \le i < j \le m \quad \text{or} \quad m+1 \le i < j \le m+n\},$$
$$\mathcal{I}_1 = \{(i,j)|1 \le i \le m < j \le m+n\}.$$

$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ has the standard basis e_{ij}, $1 \le i,j \le m+n$. We denote e_{ji} with $i<j$ also by f_{ij}. Then we get $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_1$, where

$$\mathfrak{g}_1 = \langle e_{ij}|(i,j) \in \mathcal{I}_1\rangle \quad \mathfrak{g}_{-1} = \langle f_{ij}|(i,j) \in \mathcal{I}_1\rangle.$$

Let $\mathfrak{g}^+$ be the subalgebra $\mathfrak{g}_{\bar{0}} + \mathfrak{g}_1$ of $\mathfrak{g}$. The parity of the basis elements is given by

$$\bar{e}_{ij} = \bar{f}_{ij} = \begin{cases} \bar{0}, & \text{if } (i,j) \in \mathcal{I}_0 \text{ or } i=j \\ \bar{1}, & \text{if } (i,j) \in \mathcal{I}_1. \end{cases}$$

Let $\mathfrak{H} = \langle e_{ii}|1 \le i \le m+n\rangle$. Then the set of positive roots of $\mathfrak{g}$ relative to $\mathfrak{H}$ is $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$, where

$$\Phi_0^+ = \{\epsilon_i - \epsilon_j|(i,j) \in \mathcal{I}_0\}, \Phi_1^+ = \{\epsilon_i - \epsilon_j|(i,j) \in \mathcal{I}_1\}.$$

Let $\Lambda = \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2 + \cdots + \mathbb{Z}\epsilon_{m+n} \subseteq \mathfrak{H}^*$. There is a symmetric bilinear form on $\mathfrak{H}^*$ defined by

$$(\epsilon_i, \epsilon_j) = \begin{cases} \delta_{ij}, & i \leq m \\ -\delta_{ij}, & i > m. \end{cases}$$

Let

$$\rho_0 = 1/2 \sum_{\alpha \in \Phi_0^+} \alpha, \quad \rho_1 = 1/2 \sum_{\alpha \in \Phi_1^+} \alpha,$$

and set $\rho =: \rho_0 - \rho_1 \in \Lambda$. An element $\lambda \in \mathfrak{H}^*$ is called typical if $(\lambda + \rho, \alpha) \neq 0$ for all $\alpha \in \Phi_1^+$. Defining $P(\lambda) = \Pi_{\alpha \in \Phi_1^+}(\lambda + \rho, \alpha)$ for $\lambda \in \mathfrak{H}^*$, we have $P(\lambda) \in \mathbb{Z}$ for each $\lambda \in \Lambda$. An element $\lambda \in \Lambda$ is called *p-typical* if $P(\lambda) \notin p\mathbb{Z}$.

Let $\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots + \lambda_{m+n}\epsilon_{m+n} \in \Lambda$. For each $(i,j) \in \mathcal{I}_1$, set $c(i,j) = i + j - 2m - 1$. A direct computation shows that λ is typical if and only if

$$\lambda_i + \lambda_j \neq c(i,j)$$

for all $(i,j) \in \mathcal{I}_1$, and λ is p-typical if and only if

$$\lambda_i + \lambda_j \not\equiv c(i,j) \mod p$$

for all $(i,j) \in \mathcal{I}_1$.

For any two non-negative integers $i < j$, set $[i,j] = \{i, i+1, \ldots, j\}$ as in [14]; set also $[i,j) = \{i, i+1, \ldots, j-1\}$.

Note: For each $\mu \in \mathbb{Z}^+$, there is a $\lambda = \sum \lambda_i \epsilon_i \in \Lambda$ such that $\lambda_i - \lambda_{i+1} \geq \mu$ for all $i \in [1, m+n) \setminus m$. First, let $\lambda_{m+i} = (n-i)\mu$, for $i = 1, \ldots, n$. To choose λ_i for $i \in [1, m]$, we proceed by induction on i. Let λ_m be such that $\lambda_m + \lambda_{m+i} \neq c(m, m+i)$ for all $i \in [1, n]$. Assume we have chosen λ_i for $1 < i \leq m$. Let λ_{i-1} be such that $\lambda_{i-1} + \lambda_{m+j} \neq c(i-1, m+j)$ for all $j \in [1, n]$ and $\lambda_{i-1} \geq \lambda_i + \mu$. Then we obtain $\lambda \in \Lambda$ as desired.

Put

$$h_{\alpha_i} = e_{ii} - (-1)^{\delta_{im}} e_{i+1,i+1}, e_{\alpha_i} = e_{i,i+1}, f_{\alpha_i} = e_{i+1,i}$$

for $i \in [1, m+n)$, as well as $h_{\alpha_m} = e_{m+n,m+n}$. The Cartan matrix of the Lie superalgebra $\mathfrak{g}$ is $A = (a_{ij})_{1 \leq i,j \leq m+n-1}$ with

$$a_{ij} = \begin{cases} 2, & \text{if } i = j \neq m \\ -1, & \text{if } |i - j| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

We now define the augmented Cartan matrix $\tilde{A}$. This is the $(m+n) \times (m+n-1)$ matrix whose first $m+n-1$ rows are the rows of A and whose last row is $(0, \ldots, 0, -1)$.

The Serre-type relations for the universal enveloping superalgebra of the special linear superalgebra $sl(m,n)$ are given in [19], from which one can easily show that

$\overline{U}(\mathfrak{g})$ is generated by the elements $e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_j}, i \in [1, m+n), j \in [1, m+n]$ and the relations

$$
\begin{gathered}
h_{\alpha_i} h_{\alpha_j} = h_{\alpha_j} h_{\alpha_i} \\
h_{\alpha_i} e_{\alpha_j} - e_{\alpha_j} h_{\alpha_i} = a_{ij} e_{\alpha_j}, h_{\alpha_i} f_{\alpha_j} - f_{\alpha_j} h_{\alpha_i} = -a_{ij} f_{\alpha_j}, \\
e_{\alpha_i} f_{\alpha_j} - (-1)^{\delta_{im}} f_{\alpha_j} e_{\alpha_i} = \delta_{ij} h_{\alpha_i}, \\
e_{\alpha_i} e_{\alpha_j} = e_{\alpha_j} e_{\alpha_i}, f_{\alpha_i} f_{\alpha_j} = f_{\alpha_j} f_{\alpha_i}, \quad \text{if} \quad |i-j| > 1, \\
e_{\alpha_i}^2 e_{\alpha_j} - 2 e_{\alpha_i} e_{\alpha_j} e_{\alpha_i} + e_{\alpha_j} e_{\alpha_i}^2 = 0, f_{\alpha_i}^2 f_{\alpha_j} - 2 f_{\alpha_i} f_{\alpha_j} f_{\alpha_i} + f_{\alpha_j} f_{\alpha_i}^2 = 0, \quad \text{if} \quad a_{ij} = -1, i \neq m, \\
e_{\alpha_m}^2 = f_{\alpha_m}^2 = 0, \\
[e_{\alpha_m}, [e_{\alpha_{m-1}}, [e_{\alpha_m}, e_{\alpha_{m+1}}]]] = 0, [f_{\alpha_m}, [f_{\alpha_{m-1}}, [f_{\alpha_m}, f_{\alpha_{m+1}}]]] = 0.
\end{gathered}
$$

Let $\mathfrak{g}$ be the Lie superalgebra $gl(m,n)$ over an algebraically closed field $\mathbb{F}$. Let M be a simple $\overline{U}(\mathfrak{g}_{\bar{0}})$-module with highest weight $\lambda \in \mathfrak{H}^*$. Regard M as a $\overline{U}(\mathfrak{g}^+)$-module by letting $\mathfrak{g}_1$ act trivially. Then the induced module

$$\mathcal{K}(\lambda) = \overline{U}(\mathfrak{g}) \otimes_{\overline{U}(\mathfrak{g}^+)} M$$

is called the Kac module. In case $\mathbb{F} = \mathbb{C}$, [10, Prop. 2.9] says that $\mathcal{K}(\lambda)$ is a simple $\overline{U}(\mathfrak{g})$-module if and only if λ is typical.

Let $\mathbb{F}$ be a field with char.$\mathbb{F} = 0$, and let v be an intermediate over $\mathbb{F}$. Then the quantum supergroup $U_q(gl(m,n))$(see [21, p.1237]) is defined as the $\mathbb{F}(v)$-superalgebra with the generators $K_j, K_j^{-1}, E_{i,i+1}, F_{i,i+1}$(denoted E_i^{i+1} in [21]), $i \in [1, m+n)$, and the relations

$$(1) \quad K_i K_j = K_j K_i, K_i K_i^{-1} = 1,$$

$$(2) \quad K_i E_{j,j+1} K_i^{-1} = v_i^{(\delta_{ij} - \delta_{i,j+1})} E_{j,j+1}, \quad K_i F_{j,j+1} K_i^{-1} = v_i^{-(\delta_{ij} - \delta_{i,j+1})} F_{j,j+1},$$

$$(3) \quad [E_{i,i+1}, F_{j,j+1}] = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{v_i - v_i^{-1}},$$

$$(4) \quad E_{m,m+1}^2 = F_{m,m+1}^2 = 0,$$

$$(5) \quad E_{i,i+1} E_{j,j+1} = E_{j,j+1} E_{i,i+1}, \quad F_{i,i+1} F_{j,j+1} = F_{j,j+1} F_{i,i+1}, |i-j| > 1,$$

$$(6) \quad E_{i,i+1}^2 E_{j,j+1} - (v + v^{-1}) E_{i,i+1} E_{j,j+1} E_{i,i+1} + E_{j,j+1} E_{i,i+1}^2 = 0 \quad (|i-j| = 1, i \neq m),$$

$$(7) \quad F_{i,i+1}^2 F_{j,j+1} - (v + v^{-1}) F_{i,i+1} F_{j,j+1} F_{i,i+1} + F_{j,j+1} F_{i,i+1}^2 = 0 \quad (|i-j| = 1, i \neq m),$$

$$(8) \quad [E_{m-1,m+2}, E_{m,m+1}] = [F_{m-1,m+2}, F_{m,m+1}] = 0,$$

where

$$v_i = \begin{cases} v, & \text{if } i \leq m \\ v^{-1}, & \text{if } i > m. \end{cases}$$

Most often, we shall use E_{α_i}(resp. F_{α_i}; K_{α_i}) to denote $E_{i,i+1}$(resp. $F_{i,i+1}$; $K_i K_{i+1}^{-1}$) for $\alpha_i = \epsilon_i - \epsilon_{i+1}$.

Remark: (1) For each pair of indices (i,j) with $1 \le i < j \le m+n$, the notation E_{ij}, F_{ij}(denoted respectively by E^i_j and E^j_i in [21]) are defined by

$$\begin{array}{l} E_{ij} = E_{ic}E_{cj} - v_c^{-1}E_{cj}E_{ic}, \\ F_{ij} = -v_cF_{ic}F_{cj} + F_{cj}F_{ic}, \end{array} \quad i < c < j.$$

(2) The parity of the elements E_{ij}, F_{ij} is defined by $\bar{E}_{ij} = \bar{F}_{ij} = \bar{e}_{ij} \in \mathbb{Z}_2$.

(3) The bracket product (denote $[,\}$ in [21]) in $U_q(gl(m,n))$ is defined by

$$[x,y] = xy - (-1)^{\bar{x}\bar{y}}yx.$$

Therefore the relations (8) can be written as (see [13])

$$\begin{array}{c} E_{\alpha_{m-1}}E_{\alpha_m}E_{\alpha_{m+1}}E_{\alpha_m} + E_{\alpha_m}E_{\alpha_{m-1}}E_{\alpha_m}E_{\alpha_{m+1}} + E_{\alpha_{m+1}}E_{\alpha_m}E_{\alpha_{m-1}}E_{\alpha_m} \\ +E_{\alpha_m}E_{\alpha_{m+1}}E_{\alpha_m}E_{\alpha_{m-1}} - (v+v^{-1})E_{\alpha_m}E_{\alpha_{m-1}}E_{\alpha_{m+1}}E_{\alpha_m} = 0, \end{array}$$

$$\begin{array}{c} F_{\alpha_{m-1}}F_{\alpha_m}F_{\alpha_{m+1}}F_{\alpha_m} + F_{\alpha_m}F_{\alpha_{m-1}}F_{\alpha_m}F_{\alpha_{m+1}} + F_{\alpha_{m+1}}F_{\alpha_m}F_{\alpha_{m-1}}F_{\alpha_m} \\ +F_{\alpha_m}F_{\alpha_{m+1}}F_{\alpha_m}F_{\alpha_{m-1}} - (v+v^{-1})F_{\alpha_m}F_{\alpha_{m-1}}F_{\alpha_{m+1}}F_{\alpha_m} = 0. \end{array}$$

Let $U_q(\mathfrak{g}_{\bar{0}})$ be the quantum deformation of the Lie algebra $\mathfrak{g}_{\bar{0}}$ defined by the even generators $E_{\alpha_i}, F_{\alpha_i}, i \in [1, m+n) \setminus m,\ K_j^{\pm 1}, j \in [1, m+n]$ and the relations 2.1(1)-(3), (5)-(7) involving only even generators. Then it is easy to see that the Hopf algebra $U_q(\mathfrak{g}_{\bar{0}})$ is the direct sum of its normal Hopf subalgebras:

$$U_q(\mathfrak{g}_{\bar{0}}) = U_q(gl_m) \oplus U_q(gl_n).$$

To simplify notation, we set $U_q := U_q(gl(m,n))$. According to [21, p. 1238], U_q has a unique Hopf superalgebra structure (Δ, S, ϵ) such that for all $i \in [1, m+n), j \in [1, m+n]$ that

$$\Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes K_{\alpha_i} + 1 \otimes E_{\alpha_i},$$
$$\Delta(F_{\alpha_i}) = F_{\alpha_i} \otimes 1 + K_{\alpha_i}^{-1} \otimes F_{\alpha_i},$$
$$\Delta(K_j) = K_j \otimes K_j;$$
$$S(E_{\alpha_i}) = -E_{\alpha_i}K_{\alpha_i}^{-1},$$
$$S(F_{\alpha_i}) = -K_{\alpha_i}F_{\alpha_i},$$
$$S(K_j) = K_j^{-1};$$
$$\epsilon(E_{\alpha_i}) = \epsilon(F_{\alpha_i}) = 0, \epsilon(K_j) = \epsilon(K_j^{-1}) = 1.$$

For a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we let $\mathcal{H}(V)$ denote the set of all homogeneous elements in V.

Remark: (1) Let $\mathfrak{A} = \mathfrak{A}_{\bar{0}} \oplus \mathfrak{A}_{\bar{1}}$ be a superalgebra. Then the product in $\mathfrak{A} \otimes \mathfrak{A}$ is given by

$$(a \otimes b)(c \otimes d) = (-1)^{\bar{b}\bar{c}}ac \otimes bd, a, b, c, d \in \mathcal{H}(\mathfrak{A}).$$

(2) In a Hopf superalgebra $\mathfrak{A} = \mathfrak{A}_{\bar{0}} \oplus \mathfrak{A}_{\bar{1}}$, the antipode S is a $\mathbb{Z}_2$-graded anti-automorphism, that is

$$S(u_1 u_2) = (-1)^{\bar{u}_1 \bar{u}_2} S(u_2) S(u_1),$$

for all $u_1, u_2 \in \mathcal{H}(\mathfrak{A})$.

Throughout the paper, all the subalgebras, ideals of the superalgebras to be discussed are assumed to be $\mathbb{Z}_2$-graded. Also, the modules and submodules over a superalgebra are all assumed to be $\mathbb{Z}_2$-graded. The homomorphism between two superalgebras or two modules over a superalgebra is always assumed to be even.

2.2 The braid group action on U_q

For $i \in [1, m+n) \setminus m$, the automorphism T_{α_i} of U_q is defined by(see [21, Appendix A] and also [14, 1.3])

$$T_{\alpha_i}(E_{\alpha_j}) = \begin{cases} -F_{\alpha_i} K_{\alpha_i}, & \text{if } i = j, \\ E_{\alpha_j}, & \text{if } a_{ij} = 0, \\ -E_{\alpha_i} E_{\alpha_j} + v_i^{-1} E_{\alpha_j} E_{\alpha_i}, & \text{if } a_{ij} = -1. \end{cases}$$

$$T_{\alpha_i} F_{\alpha_j} = \begin{cases} -K_{\alpha_i}^{-1} E_{\alpha_i}, & \text{if } i = j, \\ F_{\alpha_j}, & \text{if } a_{ij} = 0, \\ -F_{\alpha_j} F_{\alpha_i} + v_i F_{\alpha_i} F_{\alpha_j}, & \text{if } a_{ij} = -1. \end{cases}$$

$$T_{\alpha_i} K_j = \begin{cases} K_{i+1}, & \text{if } j = i, \\ K_i, & \text{if } j = i+1, \\ K_j, & \text{if } j \neq i, i+1. \end{cases}$$

It is pointed out in [21] that each T_{α_i} is a $\mathbb{Z}_2$-graded automorphism of U_q, which means(see [21, Appendix. A])

$$T_{\alpha_i}(uv) = (-1)^{\bar{u}\bar{v}} T_{\alpha_i}(u) T_{\alpha_i}(v), u, v \in \mathcal{H}(U_q).$$

But a straightforward computation shows that T_{α_i} is an even automorphism for U_q, that is,

$$T_{\alpha_i}(uv) = T_{\alpha_i}(u) T_{\alpha_i}(v), \quad \text{for all} \quad u, v \in \mathcal{H}(U_q).$$

In fact, one can see this by checking that $T_{\alpha_s} (s \in [1, m+n) \setminus m)$ preserves the relation 2.1(3) in the case $i = j = m$.

Assume $l \geq 1$. Let $\eta \in \mathbb{C}$ be a primitive lth root of unity. We define the $\mathbb{F}(\eta)$-superalgebra $U_\eta(gl(m,n)) = U_q/(v - \eta)U_q$. Then $U_\eta(gl(m,n))$ can also be regarded as a $\mathbb{F}(\eta)$-superalgebra defined by the same generators as those for U_q, and the same relations with each v replaced by η. For each $i \in [1, m+n) \setminus m$, we define T_{α_i}: $U_\eta(gl(m,n)) \longrightarrow U_\eta(gl(m,n))$ by the same formulas as above, with v replaced by η.

Assume $\eta^4 \neq 1$. Multiplying both sides 2.1(6)(resp. 2.1(7)) in the case $i = m-1$, $j = m$ by $E_{m,m+1}$(resp. $F_{m,m+1}$) from left and right respectively, we obtain

$$(a1) \quad E_{m-1,m}E_{m,m+1}E_{m-1,m}E_{m,m+1} - E_{m,m+1}E_{m-1,m}E_{m,m+1}E_{m-1,m} = 0,$$

$$(a2) \quad F_{m-1,m}F_{m,m+1}F_{m-1,m}F_{m,m+1} - F_{m,m+1}F_{m-1,m}F_{m,m+1}F_{m-1,m} = 0.$$

Lemma 2.1. *Assume $\eta^4 \neq 1$. Each $T_{\alpha_i}(i \in [1, m+n) \setminus m)$ is an automorphism of $U_\eta(gl(m,n))$.*

Proof. The proof follows essentially from the same computations as those for the $\mathbb{F}(v)$-superalgebra U_q. We only show that T_{α_i} preserves the relation 2.1(4). Using the relation 2.1(6), one can show that

$$(T_{\alpha_i}E_{\alpha_m})^2 = E_{\alpha_i}E_{\alpha_m}E_{\alpha_i}E_{\alpha_m} - E_{\alpha_m}E_{\alpha_i}E_{\alpha_m}E_{\alpha_i}$$

for $i = m \pm 1$. Then the lemma follows from identities (a1), (a2). $\square$

By a straightforward computation ([21, A3]), one obtains for each $i \in [1, m+n) \setminus m$ the inverse map $T_{\alpha_i}^{-1}$:

$$T_{\alpha_i}^{-1}E_{\alpha_j} = \begin{cases} -K_{\alpha_i}^{-1}F_{\alpha_i}, & \text{if } i = j, \\ E_{\alpha_j}, & \text{if } a_{ij} = 0, \\ -E_{\alpha_j}E_{\alpha_i} + v_i^{-1}E_{\alpha_i}E_{\alpha_j}, & \text{if } a_{ij} = -1. \end{cases}$$

$$T_{\alpha_i}^{-1}F_{\alpha_j} = \begin{cases} -E_{\alpha_i}K_{\alpha_i}, & \text{if } i = j, \\ F_{\alpha_j}, & \text{if } a_{ij} = 0, \\ -F_{\alpha_i}F_{\alpha_j} + v_iF_{\alpha_j}F_{\alpha_i}, & \text{if } a_{ij} = -1. \end{cases}$$

$$T_{\alpha_i}^{-1}K_j = \begin{cases} K_{i+1}, & \text{if } j = i, \\ K_i, & \text{if } j = i+1, \\ K_j, & \text{if } j \neq i, i+1. \end{cases}$$

A bijective (even)$\mathbb{F}$-linear map f from an $\mathbb{F}$-superalgebra $\mathfrak{A}$ to itself is called an anti-automorphism if $f(xy) = f(y)f(x)$ for any $x, y \in \mathcal{H}(\mathfrak{A})$.

There are $\mathbb{Z}_2$-graded algebra automorphism Ψ and antiautomorphism Ω of U_q defined by

$$\Psi(E_{\alpha_i}) = F_{\alpha_i}, \Psi(F_{\alpha_i}) = E_{\alpha_i}, \Psi(K_j) = K_j, \Psi(v) = v^{-1},$$

$$\Omega(E_{\alpha_i}) = F_{\alpha_i}, \Omega(F_{\alpha_i}) = E_{\alpha_i}, \Omega(K_j) = K_j^{-1}, \Omega(v) = v^{-1}.$$

Then according to [21], we have

$$(*) \quad \Omega T_{\alpha_i} = T_{\alpha_i}\Omega.$$

A short computation shows that

$$T_{\alpha_i}T_{\alpha_j}E_{\alpha_i}=E_{\alpha_j}, T_{\alpha_i}T_{\alpha_j}F_{\alpha_i}=F_{\alpha_j}, \quad \text{if} \quad a_{ij}=-1.$$

From this observation, together with [21, Lemma 2], one can show that

$$T_{\alpha_i}T_{\alpha_j}T_{\alpha_i}=T_{\alpha_j}T_{\alpha_i}T_{\alpha_j}, \quad \text{if } a_{ij}=-1, \quad T_{\alpha_i}T_{\alpha_j}=T_{\alpha_j}T_{\alpha_i}, \quad \text{if } a_{ij}\neq -1.$$

Let W be the Weyl group of the Lie algebra $\mathfrak{g}_{\bar{0}}$. Let $w \in W$ with a reduced expression $w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_t}}$. Then we obtain a well-defined superalgebra isomorphism $T_w : U_q \longrightarrow U_q$ such that $T_w = T_{\alpha_{i_1}} T_{\alpha_{i_2}} \cdots T_{\alpha_{i_t}}$.

Suppose $i < k < k+1 < j$. The following identities given in [21] can be verified easily by induction:

$$\begin{array}{ll}(b1) & E_{i,j}=(-1)^{j-i-1}T_{\alpha_i}T_{\alpha_{i+1}}\cdots T_{\alpha_{k-1}}T^{-1}_{\alpha_{j-1}}T^{-1}_{\alpha_{j-2}}\cdots T^{-1}_{\alpha_{k+1}}E_{k,k+1},\\ (b2) & F_{i,j}=(-1)^{j-i-1}T_{\alpha_i}T_{\alpha_{i+1}}\cdots T_{\alpha_{k-1}}T^{-1}_{\alpha_{j-1}}T^{-1}_{\alpha_{j-2}}\cdots T^{-1}_{\alpha_{k+1}}F_{k,k+1}.\end{array}$$

Applying formula $(*)$ above we get $\Omega(E_{i,j}) = F_{i,j}((i,j) \in \mathcal{I}_0 \cup \mathcal{I}_1)$. It then follows from formulas (b1), (b2) that $E^2_{ij} = F^2_{ij} = 0$ for $(i,j) \in \mathcal{I}_1$.

3 Some relations in U_q

It should be noted that there is an imprecision in [21, Lemma 2]. The formulas (see [21, (4)])

$$\begin{array}{l}[E_{a,b},E_{c,d}]=(v_b-v_b^{-1})E_{a,d}E_{c,b},\\ [F_{a,b},F_{c,d}]=-(v_b-v_b^{-1})F_{a,d}F_{c,b}\end{array} \quad a<b, c<d$$

hold provided that not just $[a,b]\cap[c,d]\neq\phi$ as pointed out in [21], but one also needs assume $[a,b]\nsubseteq[c,d]$ and $[c,d]\nsubseteq[a,b]$. Indeed, we have

Lemma 3.1. *Assume $i<s<t<j$. Then*

$$(a) \quad [E_{i,j},E_{s,t}]=0, \quad (b) \quad [F_{i,j},F_{s,t}]=0.$$

Proof. Note that (b) follows from (a) using the involution Ω, so it suffices to prove (a). We proceed by induction on $t-s$. The case $t-s=1$ is an immediate consequence of [21, 22, Lemma 1]. Now assume $t-s\geq 2$.

Recall that

$$E_{s,t}=E_{s,c}E_{c,t}-v_c^{-1}E_{c,t}E_{s,c}, s<c<t.$$

Using induction hypothesis and the identity $\bar{E}_{s,t}=\bar{E}_{s,c}+\bar{E}_{c,t}$, we get

$$\begin{aligned}[E_{i,j},E_{s,t}]&=E_{i,j}E_{s,t}-(-1)^{\bar{E}_{i,j}\bar{E}_{s,t}}E_{s,t}E_{i,j}\\ &=[E_{i,j},E_{s,c}]E_{c,t}-v_c^{-1}[E_{i,j},E_{c,t}]E_{s,c}=0.\end{aligned}$$

□

The following list of formulas will be useful.

Lemma 3.2. *[21, p. 1238-1239]*

$$(1)\quad [E_{i,j},F_{c,c+1}]=\delta_{c+1,j}E_{i,c}K_cK_{c+1}^{-1}v_c^{-1}-\delta_{i,c}(-1)^{\delta_{cm}}E_{c+1,j}K_c^{-1}K_{c+1},$$

$i<j, i\neq c,$ *or* $j\neq c+1$.

$$(2)\quad [E_{i,j},F_{i,j}]=(K_iK_j^{-1}-K_i^{-1}K_j)/(v_i-v_i^{-1}).$$

$$(3)\quad [F_{s,i},E_{s,j}]=\begin{cases}E_{i,j}K_sK_j^{-1}v_j, & i>j>s\\ E_{i,j}K_s^{-1}K_i, & j>i>s.\end{cases}$$

$$(4)\quad E_{s,i}E_{s,j}=(-1)^{\bar{E}_{s,i}}v_sE_{s,j}E_{s,i},\quad s<i<j.$$

$$(5)\quad E_{js}E_{is}=(-1)^{\bar{E}_{js}}v_s^{-1}E_{is}E_{js},\quad i<j<s.$$

$$(6)\quad [E_{i,j},F_{s,t}]=(v_j-v_j^{-1})F_{j,t}E_{i,s}K_j^{-1}K_i,\quad i\le s\le j<t.$$

It follows from formula (1) that $[E_{i,j},F_{c,c+1}]=0$, if $i<c<c+1<j$. Applying a similar proof as that for Lemma 3.1, one gets, for $i<s<t<j$,

$$(7)\quad [E_{i,j},F_{s,t}]=0,$$

and hence

$$(8)\quad [F_{i,j},E_{s,t}]=0.$$

Set

$$\mathfrak{G}^+=\{E_{ij}|(i,j)\in\mathcal{I}_0\cup\mathcal{I}_1\},\mathfrak{G}_0^+=\{E_{ij}|(i,j)\in\mathcal{I}_0\},\mathfrak{G}_1^+=\{E_{ij}|(i,j)\in\mathcal{I}_1\},$$

$$\mathcal{H}=\{K_i|1\le i\le m+n\},\quad \mathfrak{G}^-=\Omega\mathfrak{G}^+,\quad \mathfrak{G}_0^-=\Omega\mathfrak{G}_0^+,\quad \mathfrak{G}_1^-=\Omega\mathfrak{G}_1^+.$$

For $x,y\in\mathfrak{G}=:\mathfrak{G}^-\cup\mathcal{H}\cup\mathfrak{G}^+$, we write $x\prec y$ if one of the following conditions holds:

(i) $x\in\mathfrak{G}_1^-$, $y\in\mathfrak{G}_0^-\cup\mathcal{H}\cup\mathfrak{G}^+$,

(ii) $x\in\mathfrak{G}^-$, $y\in\mathcal{H}\cup\mathfrak{G}^+$,

(iii) $x\in\mathcal{H}$, $y\in\mathfrak{G}^+$,

(IV) $x\in\mathfrak{G}_0^+$ and $y\in\mathfrak{G}_1^+$,

(V) $x=E_{i,j}\in\mathfrak{G}_i^+, y=E_{s,t}\in\mathfrak{G}_i^+$, $i\in\{0,1\}$, where $i<s$ or, $i=s$ and $j<t$,

(VI) $x,y\in\mathfrak{G}_i^-$ with $\Omega(y)\prec\Omega(x)$.

For $x,y\in\mathfrak{G}$, we write $x\preccurlyeq y$ if $x\prec y$ or $x=y$. This order can be extended naturally to the set $\overline{\mathfrak{G}}=:\{x^n|x\in\mathfrak{G},n\in\mathbb{Z}^+\}$ by letting $x^n\preccurlyeq y^m$ if and only if $x\preccurlyeq y$. We call a product $x_1x_2\cdots x_n\in U_q(x_i\in\overline{\mathfrak{G}})$ a standard monomial if $x_i\preccurlyeq x_j$ whenever $i<j$.

Lemma 3.3. *Let $x, y \in \mathfrak{G}$ with $x \prec y$. Then yx is a linear combination of the standard monomials. In particular, if $x, y \in \mathfrak{G}^+$, then*

$$yx = \sum_{x \preccurlyeq x_i \prec y_i \preccurlyeq y} c_i x_i y_i, \quad c_i \in \mathbb{Z}[v, v^{-1}].$$

Proof. As the first part follows directly from the identities (1)-(6) above, we shall prove only the second part.

Let $x = E_{i,j}$, $y = E_{s,t}$. The only nontrivial verification is the case $i < s < j < t$. Using the formulas preceding Lemma 3.1, we get

$$yx = E_{s,t}E_{i,j} = (-1)^{\bar{E}_{i,j}\bar{E}_{s,t}} E_{i,j}E_{s,t} + (v_t - v_t^{-1})E_{s,j}E_{i,t}.$$

By Lemma 3.1, we have $[E_{s,j}, E_{i,t}] = 0$. Since $E_{i,j} \prec E_{i,t} \prec E_{s,j} \prec E_{s,t}$, the lemma follows immediately in case of $x, y \in \mathfrak{G}_i$, $i = 0, 1$. If $x \in \mathfrak{G}_0$ and $y \in \mathfrak{G}_1$, then we must have $i < s < j \leq m < t$, so that $E_{i,j} \prec E_{s,j} \prec E_{i,t} \prec E_{s,t}$. □

Let $E_1^{\underline{\delta}}$ denote the standard monomial $\Pi_{(i,j)\in\mathcal{I}_1} E_{i,j}^{\delta_{ij}}$, $\delta_{ij} \in \{0, 1\}$. Set $|\underline{\delta}| = \sum \delta_{ij}$. For $k \geq 0$, let $\mathcal{N}_1^{(k)} = \langle E_1^{\underline{\delta}} || \underline{\delta}| = k\rangle$. Set

$$\mathcal{N}_1 = \sum_{k\geq 0} \mathcal{N}_1^{(k)} \quad \mathcal{N}_1^+ = \sum_{k>0} \mathcal{N}_1^{(k)}.$$

Let E_{ij} be such that $(i,j) \in \mathcal{I}_1$. Suppose $E_{ij} \preccurlyeq E_{st}$. By the definition (V) of the order $\preccurlyeq$ we get $(s,t) \in \mathcal{I}_1$. It then follows from Lemma 3.3 that $\mathcal{N}_1^{(i)}\mathcal{N}_1^{(j)} \subseteq \mathcal{N}_1^{(i+j)}$; hence $(\mathcal{N}_1^+)^{nm+1} = 0$. Let

$$\mathcal{N}_{-1} =: \Omega\mathcal{N}_1, \quad \mathcal{N}_{-1}^+ = \Omega\mathcal{N}_1^+.$$

With a straightforward computation, one gets

$$U_q = \mathcal{N}_{-1}U'_q(\mathfrak{g}_{\bar{0}})\mathcal{N}_1,$$

where $U'_q(\mathfrak{g}_{\bar{0}})$ denotes the subalgebra of U_q generated by even generators. It is a homomorphic image of the quantum group $U_q(\mathfrak{g}_{\bar{0}})$. It is clear that $U'_q(\mathfrak{g}_{\bar{0}})\mathcal{N}_1$ is a subalgebra of U_q, and $U'_q(\mathfrak{g}_{\bar{0}})\mathcal{N}_1^+$ is a nilpotent ideal of $U'_q(\mathfrak{g}_{\bar{0}})\mathcal{N}_1$.

4 The PBW theorem

In this section, we give an algebraic proof of the PBW theorem for U_q.

4.1 The Hopf superalgebra structure of U_q

Let $\check{U}_q$ be a $\mathbb{F}(v)$-superalgebra defined with the generators $E_{i,i+1}, F_{i,i+1}, K_i, K_i^{-1}$ and the relations 2.1(1)-(3). Then U_q is the quotient of $\check{U}_q$. We use notation like E_{ij}, F_{ij}, etc, for the corresponding elements in $\check{U}_q$ and U_q; it will be clear from the context what is meant.

It is easy to see that

Lemma 4.1. *There are $\mathbb{Z}_2$-graded automorphism Ψ and anti-automorphism Ω of $\check{U}_q$ such that*

$$\Psi(E_{\alpha_i}) = F_{\alpha_i}, \Psi(F_{\alpha_i}) = E_{\alpha_i}, \Psi(K_i) = K_i, \Psi(v) = v^{-1},$$

$$\Omega(E_{\alpha_i}) = F_{\alpha_i}, \Omega(F_{\alpha_i}) = E_{\alpha_i}, \Omega(K_i) = K_i^{-1}, \Omega(v) = v^{-1}.$$

Define an $\mathbb{F}$-linear map $\bar{\Omega}: \quad \check{U}_q \otimes \check{U}_q \longrightarrow \check{U}_q \otimes \check{U}_q$ by setting

$$\bar{\Omega}(a \otimes b) = \Omega(b) \otimes \Omega(a).$$

Then it is easy to check that

$$(a) \quad \bar{\Omega}(u_1 u_2) = \bar{\Omega}(u_2)\bar{\Omega}(u_1), u_1, u_2 \in \mathcal{H}(\check{U}_q \otimes \check{U}_q),$$

$$(b) \quad \bar{\Omega}\Delta = \Delta\Omega.$$

Lemma 4.2. *There is on $\check{U}_q$ a unique structure (Δ, ϵ, S) of a Hopf superalgebra satisfying the same formulas as in 2.1.*

The proof of the lemma follows from a straightforward computation.

Recall the notation Λ and Φ^+. For each $\mu = l_1\epsilon_1 + \cdots + l_{m+n}\epsilon_{m+n} \in \Lambda$, set $K_\mu = \Pi_{i=1}^{m+n} K_i^{l_i} \in \check{U}_q$, so that $K_\nu = \Pi K_{\alpha_i}^{k_i}$, if $\nu = \sum k_i\alpha_i \in \mathbb{Z}\Phi^+$. For each finite sequence $I = (\alpha_1, \cdots, \alpha_r)$ of simple roots, we denote

$$E_I =: E_{\alpha_1} \cdots E_{\alpha_r}, F_I = F_{\alpha_1} \cdots F_{\alpha_r}, wtI = \alpha_1 + \cdots + \alpha_r.$$

In particular, we let $E_\phi = F_\phi = 1$.

The following lemma can be proved by a similar proof to that of [8, Lemma 4.12]. Using the formula (b) preceding Lemma 4.2, one needs check only the first identity.

Lemma 4.3. *Let I be a sequence as above. We can find elements $C^I_{A,B} \in \mathbb{Z}[v, v^{-1}]$ indexed by finite sequences of simple roots A and B with $wtI = wtA + wtB$ such that in $\check{U}_q$ and in U_q*

$$\Delta(E_I) = \sum_{A,B} C^I_{A,B}(v) E_A \otimes K_{wtA} E_B$$

$$\Delta(F_I) = \sum_{A,B} C^I_{A,B}(v^{-1}) F_A K^{-1}_{wtB} \otimes F_B.$$

We have $c^I_{A,\phi} = \delta_{A,I}$ and $c^I_{\phi,B} = \delta_{B,I}$.

Let $\nu = k_1\epsilon_1 + k_2\epsilon_2 + \cdots + k_{m+n}\epsilon_{m+n} \in \mathbb{Z}\Phi^+ \subseteq \Lambda$. Set

$$\check{U}^+_{q,\nu} = \langle x \in \check{U}^+_q | K_i x = v_i^{k_i} x K_i, i = 1, \ldots, m+n\rangle,$$

$$\check{U}^-_{q,-\nu} = \langle x \in \check{U}^-_q | K_i x = v_i^{-k_i} x K_i, i = 1, \ldots, m+n\rangle.$$

Let $\nu, \mu \in \mathbb{Z}\Phi^+$. We define $\nu \leq \mu$ if $\mu - \nu = \sum k_i\alpha_i,\ k_i \geq 0$. Then Lemma 4.3 implies for all $\mu \in \mathbb{Z}\Phi^+$, $\mu \geq 0$ that

$$\Delta(\check{U}^+_{q,\mu}) \subseteq \oplus_{0\leq\nu\leq\mu}\check{U}^+_{q,\nu} \otimes \check{U}^+_{q,\mu-\nu}K_\nu$$

$$\Delta(\check{U}^-_{q,-\mu}) \subseteq \oplus_{0\leq\nu\leq\mu}\check{U}^-_{q,-(\mu-\nu)}K_\nu^{-1} \otimes \check{U}^-_{q,-\nu}.$$

For each $i \in [1, m+n)$, set $\alpha_i^\vee = \epsilon_i - (-1)^{\delta_{im}}\epsilon_{i+1}$. Define a symmetric bilinear form on the $\mathbb{Z}$-lattice $\Lambda = \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2 + \cdots + \mathbb{Z}\epsilon_{m+n}$ by $(\epsilon_i|\epsilon_j) = \delta_{ij}$.

Let $\mathcal{M} = \mathcal{M}_{\bar{0}} \oplus \mathcal{M}_{\bar{1}}$ be a unitary free associative $\mathbb{F}(v)$-superalgebra with homogeneous generators ξ_i, $i \in [1, m+n)$. The parity of the generators is defined by $\bar{\xi}_i = \begin{cases} \bar{1}, & i = m \\ \bar{0}, & \text{otherwise.} \end{cases}$ For each $\xi_{i_1}\xi_{i_2}\cdots\xi_{i_k} \in \mathcal{M}$, put $[i_1 \cdots i_k] =: \sum_{s=1}^k \bar{\xi}_{i_s} \in \mathbb{Z}_2$.

Let $\mathbb{F}^*(v)$ denote the set of nonzero numbers in the field $\mathbb{F}(v)$. For each $\underline{\omega} = (\omega_1, \omega_2, \ldots, \omega_{m+n}) \in \mathbb{F}^*(v)^{m+n}$, define the endmorphisms $E_{i,i+1}$, $F_{i,i+1}$, $K_j(i \in [1, m+n),\ j \in [1, m+n])$ of $\mathcal{M}$ by

$$E_{i,i+1}\xi_{i_1}\cdots\xi_{i_r} = \sum_{1\leq s\leq r, i_s=i}(-1)^{[i_1\cdots i_{s-1}]\bar{E}_{i,i+1}}$$

$$\cdot\frac{\omega_i\omega_{i+1}^{-1}v_i^{-(\alpha_i^\vee|\alpha_{i_{s+1}}+\cdots+\alpha_{i_r})} - \omega_i^{-1}\omega_{i+1}v_i^{(\alpha_i^\vee|\alpha_{i_{s+1}}+\cdots+\alpha_{i_r})}}{v_i - v_i^{-1}}\xi_{i_1}\cdots\hat{\xi_{i_s}}\cdots\xi_{i_r}.$$

$$F_{i,i+1}\xi_{i_1}\cdots\xi_{i_r} = \xi_i\xi_{i_1}\cdots\xi_{i_r}.$$

$$K_i\xi_{i_1}\cdots\xi_{i_r} = \omega_i v_i^{-(\epsilon_i|\alpha_{i_1}+\cdots+\alpha_{i_r})}\xi_{i_1}\cdots\xi_{i_r}.$$

One verifies easily that $E_{i,i+1}, F_{i,i+1}, K_i$ satisfy the relations 2.1(1)-(3). Therefore, $\mathcal{M}$ is a $\check{U}_q$-module. We denote it by $\mathcal{M}(\omega)$.

For each $(\omega_1, \omega_2, \ldots, \omega_{m+n}) \in \mathbb{F}^*(v)^{m+n}$, the $\mathbb{F}(v)$-superalgebra $\mathcal{M}$ becomes also a $\check{U}_q$-module by defining

$$E_{i,i+1}\xi_{i_1}\cdots\xi_{i_r} = \xi_i\xi_{i_1}\cdots\xi_{i_r},$$

$$K_i\xi_{i_1}\cdots\xi_{i_r} = \omega_i v_i^{(\epsilon_i|\alpha_{i_1}+\cdots+\alpha_{i_r})}\xi_{i_1}\cdots\xi_{i_r},$$

$$F_{i,i+1}\xi_{i_1}\cdots\xi_{i_r}$$

$$= \sum_{1\leq j\leq r, i_j=i}(-1)^{[i_1\cdots,i_{j-1}]\overline{F}_{i,i+1}}\frac{\omega_i^{-1}\omega_{i+1}v_i^{-(\alpha_i^\vee|\alpha_{i_{j+1}}+\cdots+\alpha_{i_r})} - \omega_i\omega_{i+1}^{-1}v_i^{(\alpha_i^\vee|\alpha_{i_{j+1}}+\cdots+\alpha_{i_r})}}{v_i - v_i^{-1}}$$

$$\cdot\xi_{i_1}\cdots\hat{\xi_{i_j}}\cdots\xi_{i_r}.$$

We denote this $\check{U}_q$-module by $\mathcal{M}'(\underline{\omega})$.

With Lemma 4.3 and the $\check{U}_q$-modules $\mathcal{M}(\underline{\omega})$, $\mathcal{M}'(\underline{\omega})$, Jantzen's argument ([8, Prop. 4.16]) can be applied almost verbatim to obtain

Proposition 4.4. *The elements $F_I K_\mu E_J$ with $\mu \in \Lambda$ and I, J finite sequences of simple roots are a basis of $\check{U}_q$.*

The proposition shows that the map

$$\theta : \begin{array}{c} \check{U}_q^- \otimes \check{U}_q^0 \otimes \check{U}_q^+ \longrightarrow \check{U}_q \\ u_1 \otimes u_2 \otimes u_3 \mapsto u_1 u_2 u_3 \end{array}$$

is an isomorphism of $\mathbb{F}(v)$-vector spaces.

Set

$$\check{U}_{q,0}^- = \langle F_I \in \check{U}_q^- | \alpha_m \notin I \rangle, \check{U}_{q,1}^- = \langle F_I \in \check{U}_q^- | I = (\beta_{i_1}, \cdots, \beta_{i_r}), i_r = m \rangle.$$

It is clear that $\check{U}_{q,0}^-$(resp. $\check{U}_{q,1}^-$) is a unitary(resp. non-unitary) subalgebra of $\check{U}_q^-$. Let $\check{U}_{q,i}^+ = \Omega \check{U}_{q,i}^-, i = 0, 1$. Then we get

$$\check{U}_q^- = \check{U}_{q,1}^- \otimes \check{U}_{q,0}^-, \quad \check{U}_q^+ = \check{U}_{q,0}^+ \otimes \check{U}_{q,1}^+.$$

Set $u_{ex}^+ =: [E_{m-1,m+2}, E_{m,m+1}] \in \check{U}_q$. Given $i, j \in [1, m+n)$, set $u_{ij}^+ =:$

$$\begin{cases} E_{i,i+1}^2 E_{j,j+1} - (v + v^{-1}) E_{i,i+1} E_{j,j+1} E_{i,i+1} + E_{j,j+1} E_{i,i+1}^2, & \text{if } |i-j| = 1,\ i \neq m \\ E_{i,i+1} E_{j,j+1} - E_{j,j+1} E_{i,i+1}, & \text{if } |i-j| > 1. \end{cases}$$

Let $u_{ex}^- =: \Omega(u_{ex}^+)$ and let $u_{ij}^- =: \Omega(u_{ij}^+)$ for all $i, j \in [1, m+n)$.

Let $\mathcal{I}$(resp. $\mathcal{I}^+$; $\mathcal{I}^-$) be the two-sided ideal of $\check{U}_q$(resp. $\check{U}_q^+$; $\check{U}_q^-$) generated by the elements

$$u_{ij}^\pm, E_{m,m+1}^2, F_{m,m+1}^2, u_{ex}^\pm$$
$$(\text{resp.} u_{ij}^+, E_{m,m+1}^2, u_{ex}^+; u_{ij}^-, F_{m,m+1}^2, u_{ex}^-.)$$

By a straightforward computation, one gets

Lemma 4.5.

$$\Delta(E_{m,m+1}^2) = E_{m,m+1}^2 \otimes K_m^2 K_{m+1}^{-2} + 1 \otimes E_{m,m+1}^2,$$

$$\Delta(u_{ij}^+) = \begin{cases} u_{ij}^+ \otimes K_{\alpha_i}^2 K_{\alpha_j} + 1 \otimes u_{ij}^+, & \text{if } |i-j| = 1, i \neq m \\ u_{ij}^+ \otimes K_{\alpha_i} K_{\alpha_j} + 1 \otimes u_{ij}^+, & \text{if } |i-j| > 1, \end{cases}$$

$$\Delta(u_{ex}^+) = u_{ex}^+ \otimes K_{m-1} K_m K_{m+1}^{-1} K_{m+2}^{-1} + 1 \otimes u_{ex}^+ \quad (mod \quad \mathcal{I} \otimes \check{U}_q + \check{U}_q \otimes \mathcal{I}).$$

$$S(E_{m,m+1}^2) = E_{m,m+1}^2 K_m^{-2} K_{m+1}^2,$$

$$S(u_{ij}^+) = \begin{cases} -u_{ij}^+ K_{\alpha_i}^{-2} K_{\alpha_j}^{-1}, & \text{if } |i-j| = 1, i \neq m \\ -u_{ij}^+ K_{\alpha_i}^{-1} K_{\alpha_j}^{-1}, & \text{if } |i-j| > 1. \end{cases}$$

It follows from the lemma that

$$\Delta(\mathcal{I}) \subseteq \check{U}_q \otimes \mathcal{I} + \mathcal{I} \otimes \check{U}_q, S(\mathcal{I}) \subseteq \mathcal{I}, \epsilon(\mathcal{I}) = 0.$$

It is easy to check that, for all $s \in [1, m+n)$,

$$[F_{s,s+1}, u_{ij}^+] = 0, \quad [F_{s,s+1}, E_{m,m+1}^2] = 0$$

$$[E_{s,s+1}, u_{ij}^-] = 0, \quad [E_{s,s+1}, F_{m,m+1}^2] = 0.$$

By a straightforward computation, we get

Lemma 4.6. *The following identities hold in* $\check{U}_q$.

$$[F_{m,m+1}, E_{m-1,m+2}] = 0, \quad E_{m,m+1}E_{m,m+2} + v_m E_{m,m+2}E_{m,m+1} = 0(mod \quad \mathcal{I}),$$

$$E_{m,m+1}E_{m-1,m+1} + v_{m+1}^{-1} E_{m-1,m+1}E_{m,m+1} = 0(mod \quad \mathcal{I}).$$

Applying the lemma, one verifies easily that

$$[F_{m-1,m}, u_{ex}^+] = [F_{m+1,m+2}, u_{ex}^+] = [F_{m,m+1}, u_{ex}^+] = 0(\mathrm{mod} \quad \mathcal{I}).$$

By applying Ω, one gets

$$[E_{m-1,m}, u_{ex}^-] = [E_{m+1,m+2}, u_{ex}^-] = [E_{m,m+1}, u_{ex}^-] = 0(\mathrm{mod} \quad \mathcal{I}).$$

It follows that

$$\mathcal{I} = \theta(\check{U}_q^- \otimes \check{U}_q^0 \otimes \mathcal{I}^+ + \mathcal{I}^- \otimes \check{U}_q^0 \otimes \check{U}_q^+).$$

Then we get an induced vector space isomorphism

$$U_q = \check{U}_q/\mathcal{I} \cong \bar{\theta}(\check{U}_q^-/\mathcal{I}^- \otimes \check{U}_q^0 \otimes \check{U}_q^+/\mathcal{I}^+)$$

such that $\bar{\theta}(\check{U}_q^0) = U_q^0$.

By a similar argument as that for [8, 4.11, 4.21], one gets the following result.

Corollary 4.7. *(i)* U_q *has a unique structure* (Δ, ϵ, S) *of a Hopf superalgebra given in 2.1.*

(ii) The multiplication map

$$\bar{\theta} : U_q^- \otimes U_q^0 \otimes U_q^+ \to U_q, u_1 \otimes u_2 \otimes u_3 \mapsto u_1 u_2 u_3$$

is an isomorphism of vector spaces.

(iii) The superalgebra U_q^+ *is isomorphic to the superalgebra with generators* $E_{i,i+1}$, $i \in [1, m+n)$ *and relations*

$$u_{ij}^+ = 0, E_{m,m+1}^2 = 0, u_{ex}^+ = 0.$$

(IV) The superalgebra U_q^- *is isomorphic to the superalgebra with generators* $F_{i,i+1}$, $i \in [1, m+n)$ *and relations*

$$u_{ij}^- = 0, F_{m,m+1}^2 = 0, u_{ex}^- = 0.$$

(V) The K_μ *with* $\mu \in \Lambda$ *are a basis of* U_q^0.

Recall the subalgebra $U_q'(\mathfrak{g}_{\bar 0})$ of U_q in Sec. 3. Then we have

Proposition 4.8. *There is an isomorphism of $\mathbb{F}(v)$-algebras: $U_q(\mathfrak{g}_{\bar 0}) \cong U_q'(\mathfrak{g}_{\bar 0})$.*

Proof. Let us denote by $U_q'^-$(resp. $U_q'^+$) the image of $\check{U}_{q,0}^-$(resp. $\check{U}_{q,0}^+$) in $U_q^- = \check{U}_q^-/\mathcal{I}^-$(resp. $U_q^+ = \check{U}_q^+/\mathcal{I}^+$). Then by the definition of $U_q'(\mathfrak{g}_{\bar 0})$ and Coro. 4.7(ii), we have

$$U_q'(\mathfrak{g}_{\bar 0}) = \bar\theta(U_q'^- \otimes U_q^0 \otimes U_q'^+)).$$

Let

$$U_q(\mathfrak{g}_{\bar 0}) \cong U_q(\mathfrak{g}_{\bar 0})^- \otimes U_q^0 \otimes U_q(\mathfrak{g}_{\bar 0})^+$$

be the triangular decomposition of the quantum group $U_q(\mathfrak{g}_{\bar 0})$. Using Prop. 4.4, we obtain

$$\mathcal{I}^- \cap \check{U}_{q,0}^- = \sum_{m\notin\{i,j\}} \check{U}_{q,0}^- u_{ij}^- \check{U}_{q,0}^- \quad \mathcal{I}^+ \cap \check{U}_{q,0}^+ = \sum_{m\notin\{i,j\}} \check{U}_{q,0}^+ u_{ij}^+ \check{U}_{q,0}^+.$$

It follows that

$$U_q'^- \cong \check{U}_{q,0}^-/(\mathcal{I}^- \cap \check{U}_{q,0}^-) \cong U_q(\mathfrak{g}_{\bar 0})^-$$

and similarly $U_q'^+ \cong U_q(\mathfrak{g}_{\bar 0})^+$. This establishes the proposition. □

For brevity, we shall write $U_q'(\mathfrak{g}_{\bar 0})$ also as $U_q(\mathfrak{g}_{\bar 0})$ in the following.

Recall the subalgebras $\mathcal{N}_{\pm 1}$ of U_q. Let $\pi : \check{U}_q \longrightarrow U_q$ be the canonical surjective mapping. Using the arguments in Sec.3, we get $\pi(\check{U}_{q,1}^{\pm}) \subseteq U_q(\mathfrak{g}_{\bar 0})\mathcal{N}_{\pm 1}$.

4.2 Highest weight modules for U_q

In this subsection, we construct simple highest weight U_q-modules following the procedure in [17].

For each $\mu = \mu_1\epsilon_1 + \mu_2\epsilon_2 + \cdots + \mu_{m+n}\epsilon_{m+n} \in \Lambda$, set $v^\mu = (v_1^{\mu_1}, \cdots, v_{m+n}^{\mu_{m+n}}) \in \mathbb{F}^*(v)^{m+n}$. Let $M = M_{\bar 0} \oplus M_{\bar 1}$ be a U_q-module. For any $\underline{\omega} = (\omega_1, \cdots, \omega_{m+n}) \in \mathbb{F}^*(v)^{m+n}$, set

$$(M_{\underline{\omega}})_{\bar j} = \{x \in M_{\bar j} | K_i x = \omega_i x, i = 1, \dots, m+n\}, \quad \bar j \in \mathbb{Z}_2,$$

$$M_{\underline{\omega}} = \{x \in M | K_i x = \omega_i x, i = 1, \dots, m+n\}.$$

Then it is easy to check that $M_{\underline{\omega}} = (M_{\underline{\omega}})_{\bar 0} \oplus (M_{\underline{\omega}})_{\bar 1}$ and $\sum_{\underline{\omega}} M_{\underline{\omega}}$ is a direct sum.

By the relation 2.1(2), one gets, for all $i \in [1, m+n)$,

$$E_{\alpha_i} M_{\underline{\omega}} \subseteq M_{\underline{\omega} v^{\alpha_i}}, \quad F_{\alpha_i} M_{\underline{\omega}} \subseteq M_{\underline{\omega} v^{-\alpha_i}}.$$

Specifically, one has

$$E_{\alpha_i}(M_{\underline{\omega}})_{\bar j} \subseteq (M_{\underline{\omega} v^{\alpha_i}})_{\bar j + \bar\delta_{im}}, \quad F_{\alpha_i}(M_{\underline{\omega}})_{\bar j} \subseteq (M_{\underline{\omega} v^{-\alpha_i}})_{\bar j + \bar\delta_{im}}.$$

For $\underline{\omega}_1, \underline{\omega}_2 \in \mathbb{F}^*(v)^{m+n}$, we define $\underline{\omega}_2 \leq \underline{\omega}_1$ to mean that

$$\underline{\omega}_1 \underline{\omega}_2^{-1} = v^{\sum l_i \alpha_i}, l_i \in \mathbb{N}.$$

Then one has (see [17, 2.5.2])

$$E_{\alpha_i} M_{\underline{\omega}_1} \subseteq M_{\underline{\omega}_2}, \underline{\omega}_2 \geq \underline{\omega}_1; \quad F_{\alpha_i} M_{\underline{\omega}_1} \subseteq M_{\underline{\omega}_2}, \underline{\omega}_2 < \underline{\omega}_1.$$

A homogeneous nonzero element $x \in M_{\underline{\omega}}$ is called maximal if $E_{\alpha_i} x = 0$ for all $i \in [1, m+n)$.

In the $\check{U}_q$-module $\mathcal{M}(\underline{\omega})$ given earlier, set

$$\phi_{ij} =: \begin{cases} \xi_i^2 \xi_j - (v + v^{-1}) \xi_i \xi_j \xi_i + \xi_j \xi_i^2, & \text{if } |i-j| = 1, i \neq m \\ \xi_i \xi_j - \xi_j \xi_i, & \text{if } |i-j| > 1, \end{cases}$$

$$\phi_m =: \xi_m^2, \quad \phi_{ex} =: \xi_{m-1}\xi_m\xi_{m+1}\xi_m + \xi_m\xi_{m-1}\xi_m\xi_{m+1} + \xi_{m+1}\xi_m\xi_{m-1}\xi_m + \xi_m\xi_{m+1}\xi_m\xi_{m-1} \\ -(v+v^{-1})\xi_m\xi_{m-1}\xi_{m+1}\xi_m.$$

Let $\mathcal{J}$ be the two-sided ($\mathbb{Z}_2$-graded)ideal of $\mathcal{M}(\underline{\omega})$ generated by these elements. In view of the proof of [17, Lemma 2.3], one can show that $\mathcal{J}$ is stable under the endomorphisms $E_{i,i\pm 1}, K_i^{\pm 1}$ defined earlier. With only a few modifications of [17, 2.4-2.6], one sees that the quotient algebra $\mathcal{M}^{\underline{\omega}} =: \mathcal{M}(\underline{\omega})/\mathcal{J}$ defines a U_q-module and also

Lemma 4.9. *For each $\underline{\omega} \in \mathbb{F}^*(v)^{m+n}$, there exists a simple U_q-module of highest weight $\underline{\omega}$. It is unique up to isomorphism and contains a unique maximal vector.*

A U_q-module $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is integrable if $M = \sum_{\underline{\omega}} M_{\underline{\omega}}$ and if $E_{i,i+1}, F_{i,i+1} (i \in [1, m+n) \setminus m)$ act locally nilpotently on M. Assume $i \neq m$. Using the imbedding of $\mathbb{F}$-algebras: $f : U_q(sl_2) \longrightarrow U_q$ with

$$f(E) = E_{i,i+1}, f(F) = F_{i,i+1}, f(K) = K_i K_{i+1}^{-1}, f(v) = v_i,$$

with a few modifications of the proof of [17, Prop. 3.2], one gets

Corollary 4.10. *The U_q-module $M = M_{\bar{0}} \oplus M_{\bar{1}}$ with highest weight*

$$\underline{\omega} = (\omega_1, \cdots, \omega_{m+n}) \in \mathbb{F}^*(v)^{m+n}$$

is integrable if and only if for any $i \in [1, m+n) \setminus m$, there is $l_i \in \mathbb{N}$ and $\delta_i \in \{1, -1\}$ such that

$$\omega_i \omega_{i+1}^{-1} = \delta_i v_i^{l_i}.$$

Let M be as in the corollary. For convenience, set $\delta_m = 1$. M is then called a U_q-module of type $\underline{\delta} = (\delta_1, \cdots, \delta_{m+n-1})$. Define an automorphism σ of U_q by means of

$$\sigma(E_{i,i+1}) = \delta_i E_{i,i+1}, \sigma(F_{i,i+1}) = F_{i,i+1},$$

$$\sigma(K_i) = \begin{cases} \delta_i K_i, & \text{if } i \leq m, \\ K_{m+1}, & \text{if } i = m+1, \\ \delta_{i-1}\delta_{i-2} K_i, & \text{if } i > m+1. \end{cases}$$

Twisting the U_q-module M with σ, we get a U_q-module M^σ of type $\underline{1} = (1, \cdots, 1)$. In what follows, we restrict our discussion only to the U_q-modules of type $\underline{1}$.

4.3 The superalgebra $U_{\mathcal{A}}$

Recall the notion $K_{\alpha_i} =: K_i K_{i+1}^{-1}$, $i \in [1, m+n)$. To unify notation, put $K_{\alpha_{m+n}} =: K_{m+n}$. Set

$$\begin{bmatrix} K_{\alpha_i}; c \\ t \end{bmatrix} = \Pi_{s=1}^{t} \frac{K_{\alpha_i} v_i^{c-s+1} - K_{\alpha_i}^{-1} v_i^{-c+s-1}}{v_i^s - v_i^{-s}}$$

for each $i \in [1, m+n]$, $c \in \mathbb{Z}$, $t \in \mathbb{N}$. For each $l \in \mathbb{N}$, set

$$[l] =: \frac{v^l - v^{-l}}{v - v^{-1}}.$$

Let $\mathcal{A} = \mathbb{F}[v, v^{-1}] \subseteq \mathbb{F}(v)$, and let $U_{\mathcal{A}}$(resp. $U_{\mathcal{A}}^+$; $U_{\mathcal{A}}^-$; $U_{\mathcal{A}}^0$) be the $\mathcal{A}$-subalgebra(with 1) of U_q generated by the homogeneous elements

$$E_{\alpha_i}^{(l)} = [l]!^{-1} E_{\alpha_i}^l, F_{\alpha_i}^{(l)} = [l]!^{-1} F_{\alpha_i}^l, K_{\alpha_j}^{\pm 1}, \begin{bmatrix} K_{\alpha_j}; c \\ t \end{bmatrix}$$

(resp. $E_{\alpha_i}^{(l)}$; $F_{\alpha_i}^{(l)}$; $K_{\alpha_j}^{\pm 1}$, $\begin{bmatrix} K_{\alpha_j}; c \\ t \end{bmatrix}$), $i \in [1, m+n)$, $j \in [1, m+n]$. By the formula [17, 4.3.1], we obtain $\begin{bmatrix} K_{\alpha_i}; c \\ t \end{bmatrix} \in U_{\mathcal{A}}$($i \in [1, m+n) \setminus m$, $c \in \mathbb{Z}$, $t \in \mathbb{N}$).

By a short computation, we get, for $i \in [1, m+n]$, $j \in [1, m+n)$,

$$(h1) \quad \begin{bmatrix} K_{\alpha_i}; c \\ t \end{bmatrix} E_{\alpha_j}^{(l)} = E_{\alpha_j}^{(l)} \begin{bmatrix} K_{\alpha_i}; c + l a_{ij} \\ t \end{bmatrix},$$

$$(h2) \quad \begin{bmatrix} K_{\alpha_i}; c \\ t \end{bmatrix} F_{\alpha_j}^{(l)} = F_{\alpha_j}^{(l)} \begin{bmatrix} K_{\alpha_i}; c - l a_{ij} \\ t \end{bmatrix}.$$

Recall that $E_{ij}^2 = 0$ in case $\bar{E}_{i,j} = \bar{1}$. So we denote $E_{i,j}^{(N)} = [N]!^{-1} E_{ij}^N$ only for $N = 0, 1$ if $\bar{E}_{i,j} = \bar{1}$ but for all $N \in \mathbb{N}$ if $\bar{E}_{ij} = \bar{0}$. Using formulas from Sec 3, with induction on $N, M \in \mathbb{N}$, we obtain

Lemma 4.11.

$$(1) \quad E_{i,j}^{(N)} E_{i,j}^{(M)} = \begin{bmatrix} M+N \\ N \end{bmatrix} E_{i,j}^{(N+M)};$$

$$(2) \quad E_{i,j}^{(N)} E_{s,t}^{(M)} = (-1)^{\bar{E}_{ij} \bar{E}_{st}} E_{s,t}^{(M)} E_{i,j}^{(N)},$$

for $i < s < t < j$ or $s < t < i < j$;

$$(3) \quad E_{t,a}^{(N)} E_{t,b}^{(M)} = [(-1)^{\bar{E}_{t,a}} v_t]^{NM} E_{t,b}^{(M)} E_{t,a}^{(N)}, \quad t < a < b;$$

$$(4) \quad E_{b,t}^{(N)} E_{a,t}^{(M)} = [(-1)^{\bar{E}_{b,t}} v_t^{-1}]^{NM} E_{a,t}^{(M)} E_{b,t}^{(N)}, \quad a < b < t;$$

$$(5) \quad E_{i,c}^{(N)} E_{c,j}^{(M)} = \sum_{0 \le k \le min\{N,M\}} v_c^{-(M-k)(N-k)} E_{c,j}^{(M-k)} E_{i,j}^{(k)} E_{i,c}^{(N-k)}, \quad i < c < j.$$

$$(6) \quad E_{i,i+1}^{(M)} F_{j,j+1}^{(N)} = F_{j,j+1}^{(N)} E_{i,i+1}^{(M)} \quad \text{if} \quad i \neq j.$$

With only minor adjustments of the Kac's formula in [17, 4.3], one obtains

$$(7)\quad E_{\alpha_i}^{(N)}F_{\alpha_i}^{(M)}=\sum_{0\le t\le min\{M,N\}}(-1)^{\delta_{im}(t-1)}F_{\alpha_i}^{(M-t)}\begin{bmatrix}K_{\alpha_i};2t-N-M\\ t\end{bmatrix}_i E_{\alpha_i}^{(N-t)}.$$

Using the formulas (1)-(7) above, together with (h1), (h2), we get $U_{\mathcal{A}}=U_{\mathcal{A}}^{-}U_{\mathcal{A}}^{0}U_{\mathcal{A}}^{+}$.

4.4 The proof of the PBW theorem

Let $L=L_{\bar 0}\oplus L_{\bar 1}$ be a simple U_q-module with highest weight $\underline{\omega}=(v_1^{r_1},\dots,v_{m+n}^{r_{m+n}})$ ($(r_1,\dots,r_{m+n})\in\mathbb{Z}^{m+n}$) such that $r_i-r_{i+1}\in\mathbb{N}$ for all $i\in[1,m+n)\setminus m$. Let $x\in L_{\underline{\omega}}$ be a maximal vector and set $L_{\mathcal{A}}=U_{\mathcal{A}}^{-}x\subseteq L$. Then by a proof similar to that of [17, Prop. 4.2], one gets

Lemma 4.12. *(a) $L_{\mathcal{A}}$ is a $U_{\mathcal{A}}$-submodule of L.*

(b) $\mathbb{F}(v)\otimes_{\mathcal{A}}L_{\mathcal{A}}\to L$ is an (even) isomorphism of $\mathbb{F}(v)$-vector spaces.

(c) $L_{\mathcal{A}}$ is the direct sum of $L_{\mathcal{A}}\cap L_{\underline{\omega}'}$ with each $L_{\mathcal{A}}\cap L_{\underline{\omega}'}$ is a finite generated free $\mathcal{A}$-module of finite rank.

From the discussion in Sec.3, we have

$$U_q(\mathfrak{g}_{\bar 0})\mathcal{N}_1=U_q(\mathfrak{g}_{\bar 0})+U_q(\mathfrak{g}_{\bar 0})\mathcal{N}_1^{+}$$

and $U_q(\mathfrak{g}_{\bar 0})\mathcal{N}_1^{+}$ is a nilpotent ($\mathbb{Z}_2$-graded) ideal of $U_q(\mathfrak{g}_{\bar 0})\mathcal{N}_1$. Since $U_q(\mathfrak{g}_{\bar 0})$ contains no zero divisors, the sum is direct. Let $M=M_{\bar 0}\oplus M_{\bar 1}$ be a simple $U_q(\mathfrak{g}_{\bar 0})\mathcal{N}_1$-module. Then $U_q(\mathfrak{g}_{\bar 0})\mathcal{N}_1^{+}M$ is a graded submodule of M, hence we get $U_q(\mathfrak{g}_{\bar 0})\mathcal{N}_1^{+}M=0$. Therefore each simple $U_q(\mathfrak{g}_{\bar 0})\mathcal{N}_1$-module is a simple $U_q(\mathfrak{g}_{\bar 0})$-module annihilated by $U_q(\mathfrak{g}_{\bar 0})\mathcal{N}_1^{+}$.

Let M be a simple $U_q(\mathfrak{g}_{\bar 0})$-module of highest weight $\underline{\omega}$. Regard M as a $U_q(\mathfrak{g}_{\bar 0})\mathcal{N}_1$-module annihilated by $U_q(\mathfrak{g}_{\bar 0})\mathcal{N}_1^{+}$. Define the Kac-module

$$\mathcal{K}(\underline{\omega})=U_q\otimes_{U_q(\mathfrak{g}_{\bar 0})\mathcal{N}_1}M.$$

Now let $v^+\in M_{\underline{\omega}}$ be a maximal vector. We regard $\mathbb{F}$ as a $\mathcal{A}$-module by letting v act as multiplication by 1. Set

$$\mathcal{K}_{\mathcal{A}}=U_{\mathcal{A}}v^+\subseteq\mathcal{K}(\underline{\omega}),\quad \bar{\mathcal{K}}(\underline{\omega})=\mathbb{F}\otimes_{\mathcal{A}}\mathcal{K}_{\mathcal{A}}.$$

Let $e_{i,i+1},f_{i,i+1},h_{\alpha_i},i\in[1,m+n),h_{\alpha_{m+n}},\bar{K}_j,j\in[1,m+n]$ denote respectively the endomorphisms of $\bar{\mathcal{K}}(\omega)$ induced respectively by the elements

$$E_{i,i+1},F_{i,i+1},\begin{bmatrix}K_{\alpha_i};0\\ 1\end{bmatrix},\begin{bmatrix}K_{m+n};0\\ 1\end{bmatrix},K_j.$$

Lemma 4.13. *(1)* $\bar{K}_j = 1$.

(2) Let $\mathfrak{g} = gl(m,n)$*. The elements* $e_{ij}, f_{ij}, h_{\alpha_i}$ *satisfy the relations for the universal enveloping algebra* $\overline{U}(\mathfrak{g})$*.*

(3) The element h_{α_i} *acts on* $\bar{\mathcal{K}}(\underline{\omega})_{\underline{\omega}}$ *(*$\underline{\omega} = (v_1^{r_1}, \ldots, v_{m+n}^{r_{m+n}})$*) as multiplication by*

$$\begin{cases} r_i - (-1)^{\delta_{im}} r_{i+1}, & i \in [1, m+n) \\ r_{m+n}, & i = m+n. \end{cases}$$

(4) Assume $\underline{\omega} = (v_1^{r_1}, \cdots, v_{m+n}^{r_{m+n}})$ *with* $r_i - r_{i+1} \in \mathbb{N}$ *for all* $i \in [1, m+n) \setminus m$*. Let* $\lambda = \sum_{i=1}^{m+n} r_i \epsilon_i \in \Lambda$*. Then* $\bar{\mathcal{K}}(\underline{\omega})$ *is a homomorphic image of the Kac module* $\mathcal{K}(\lambda)$*.*

Proof. The arguments leading to [17, 4.11] can be repeated to give (1)-(3).

(4) Let $v^+ \in \mathcal{K}(\underline{\omega})$ be a maximal vector. Since

$$\mathcal{K}_{\mathcal{A}} = U_{\mathcal{A}} v^+ = \mathcal{N}_{-1,\mathcal{A}} U_q(\mathfrak{g}_{\bar{0}})_{\mathcal{A}} \mathcal{N}_{1,\mathcal{A}} v^+ = \mathcal{N}_{-1,\mathcal{A}} U_q(\mathfrak{g}_{\bar{0}})_{\mathcal{A}} v^+,$$

we have

$$\bar{\mathcal{K}}(\underline{\omega}) = \wedge(\mathfrak{g}_{-1}) \overline{U}(\mathfrak{g}_{\bar{0}}) \bar{v}^+ = \wedge(\mathfrak{g}_{-1}) \overline{U}(sl_m \oplus sl_n) \bar{v}^+.$$

By the assumption on $\underline{\omega}$, the $\overline{U}(\mathfrak{g}_{\bar{0}})$-submodule $\overline{U}(\mathfrak{g}_{\bar{0}})\bar{v}^+$ is integrable. Then it is a semisimple $\overline{U}(sl_m \oplus sl_n)$-module by [11, 10.7]. Since it is generated by a unique maximal vector $\bar{v}^+$, it is a simple $\overline{U}(sl_m \oplus sl_n)$-module as noted in [17, 4.12] so that it is also a simple $\overline{U}(\mathfrak{g}_{\bar{0}})$-module. Thus, $\bar{\mathcal{K}}(\underline{\omega})$ is the epimorphic image of the Kac module $\mathcal{K}(\lambda)$. ☐

Recall the notion $E_1^{\underline{\delta}}$. For each $\underline{\psi} \in \mathbb{N}^{|\mathcal{I}_0|}$, let $E_0^{\underline{\psi}}$ denote the product $\Pi_{(i,j)\in\mathcal{I}_0} E_{ij}^{\psi_{ij}}$ in the order given in Sec. 3. Set $F_1^{\underline{\delta}} = \Omega(E_1^{\underline{\delta}}) \in U_q$ for all $\underline{\delta}$, and set $F_0^{\underline{\psi}} = \Omega(E_0^{\underline{\psi}}) \in U_q$ for all $\underline{\psi}$.

Theorem 4.14. *The set of elements*

$$\mathcal{S} = \{F_1^{\underline{\delta}} F_0^{\underline{\psi}} | \underline{\delta} \in \{0,1\}^{|\mathcal{I}_1|}, \underline{\psi} \in \mathbb{N}^{|\mathcal{I}_0|}\}$$

is a $\mathbb{F}(v)$*-basis of the subalgebra* U_q^-*.*

Proof. It suffices to show that the set $\mathcal{S} \subseteq U_q^-$ is linearly independent. It's no loss of generality to assume $\mathbb{F} = \mathbb{C}$. Let $\underline{\omega}$ and $\bar{\mathcal{K}}(\underline{\omega})$ be as in Lemma 4.13. Recall the notation $\mathcal{K}(\lambda)$. If $\lambda = r_1\epsilon_1 + \cdots + r_{m+n}\epsilon_{m+n} \in \Lambda$ is typical, so that $\mathcal{K}(\lambda)$ is simple by [10, Prop. 2.9], Lemma 4.13(4) then shows that $\bar{\mathcal{K}}(\underline{\omega}) \cong \mathcal{K}(\lambda)$.

Let $\mathcal{S}_1$ be any finite subset of $\mathcal{S}$. Choose an integer $\mu \geq 0$ such that $\psi_{ij} \leq \mu$ for all $(i,j) \in \mathcal{I}_0$ and all $\underline{\psi}$ with $F_1^{\underline{\delta}} F_0^{\underline{\psi}} \in \mathcal{S}_1$. By the representation theory of $\overline{U}(\mathfrak{g}_{\bar{0}})$, there is a finite dimensional simple $\overline{U}(\mathfrak{g}_{\bar{0}})$-module with the highest weight

$\underline{\lambda} =: (\lambda_1, \ldots, \lambda_{m+n}) \in \mathbb{Z}^{m+n}$ such that $\mu \le \lambda_i - \lambda_{i+1}$ for all $i \in [1, m+n) \setminus m$. According to the note in Sec. 2.1, we can choose $\underline{\lambda}$ to be typical.

Let $\underline{\omega} = (v_1^{\lambda_1}, \cdots, v_{m+n}^{\lambda_{m+n}})$. Then the elements of $\mathcal{S}_1$ induce linearly independent endomorphisms of the Kac module $\mathcal{K}(\lambda) \cong \bar{\mathcal{K}}(\omega)$. Thus, the set $\mathcal{S}_1$ is linearly independent, so is the set $\mathcal{S}$. $\square$

Using the theorem, combined with Coro. 4.7(ii), one obtains

Corollary 4.15. *(PBW theorem) The set of elements*

$$\{F_1^{\underline{\delta}'} F_0^{\underline{\psi}'} K_\mu E_0^{\underline{\psi}} E_1^{\underline{\delta}} | \underline{\delta}, \underline{\delta}' \in \{0,1\}^{\mathcal{I}_1}, \underline{\psi}, \underline{\psi}' \in \mathbb{N}^{\mathcal{I}_0}, \mu \in \Lambda\}$$

is a $\mathbb{F}(v)$-basis of U_q.

Consequently, we get an isomorphism of vector spaces:

$$\begin{array}{l} \mathcal{N}_{-1} \otimes U_q(\mathfrak{g}_{\bar{0}}) \otimes \mathcal{N}_1 \longrightarrow U_q \\ u^- \otimes u_0 \otimes u^+ \longrightarrow u^- u_0 u^+ \end{array}, \quad u^\pm \in \mathcal{N}_{\pm 1}, u_0 \in U_q(\mathfrak{g}_{\bar{0}}).$$

5 The $\mathcal{A}$-superalgebra U

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, and let $\mathcal{A}' = \mathbb{Q}(v)$ be its quotient field. Let $U_{\mathcal{A}'}$ be the quantum superalgebra $U_q(gl(m,n))$ over $\mathcal{A}'$(see Sec.2.). We define U(resp. U^+; U^-; U^0) to be the $\mathcal{A}$-sub-superalgebra of $U_{\mathcal{A}'}$ generated by the homogeneous elements

$$E_{i,i+1}^{(N)}, F_{i,i+1}^{(N)}, K_j^{\pm 1}, \begin{bmatrix} K_{\alpha_j}; c \\ t \end{bmatrix} (\text{resp.} \quad E_{i,i+1}^{(N)}; F_{i,i+1}^{(N)};$$

$$K_j^{\pm 1}, \begin{bmatrix} K_{\alpha_j}; c \\ t \end{bmatrix}), i \in [1, m+n), j \in [1, m+n], N \ge 0, c \in \mathbb{Z}, t \in \mathbb{N}.$$

Following [14], we shall give a description of the $\mathcal{A}$-superalgebra U in terms of generators and relations.

5.1 The superalgebra $\mathcal{V}$

In this subsection we shall introduce an $\mathcal{A}$-superalgebra $\mathcal{V}$ defined by generators and relations. Then we show that this superalgebra is isomorphic to U.

We shall consider the set consisting of the following variables:

$$\begin{array}{ll} (a) & E_{ij}^{(N)} \\ (b) & F_{ij}^{(N)} \end{array} \quad ((i,j) \in \mathcal{I}_0 \cup \mathcal{I}_1, N \in \begin{cases} \mathbb{N}, & \text{if } (i,j) \in \mathcal{I}_0 \\ \{0,1\}, & \text{if } (i,j) \in \mathcal{I}_1 \end{cases}),$$

$$(c) \quad K_{\alpha_i}, K_{\alpha_i}^{-1}, \begin{bmatrix} K_{\alpha_i}; c \\ t \end{bmatrix} \quad (i \in [1, m+n], c \in \mathbb{Z}, t \in \mathbb{N}).$$

We denote the variable $E^{(N)}_{i,i+1}$(resp. $F^{(N)}_{i,i+1}$) also by $E^{(N)}_{\alpha_i}$(resp. $F^{(N)}_{\alpha_i}$), $i \in [1, m+n)$.

Let $\mathcal{V}^+$ be the $\mathcal{A}$-superalgebra defined by the homogeneous generators (a) with the relations Lemma 4.11 (1)-(5) together with $E^{(0)}_{ij} = 1$, $E^{(2)}_{ij} = 0$ for all $(i,j) \in \mathcal{I}_1$.

Let $\mathcal{V}^-$ be the $\mathcal{A}$-superalgebra defined by the homogeneous generators (b) and the relations

$$(f1) \quad F^{(2)}_{ij} = 0, (i,j) \in \mathcal{I}_1,$$

$$(f2) \quad F^{(N)}_{ij} F^{(M)}_{ij} = \begin{bmatrix} M+N \\ N \end{bmatrix} F^{(N+M)}_{i,j}, \quad F^{(0)}_{i,j} = 1,$$

$$(f3) \quad F^{(N)}_{i,j} F^{(M)}_{s,t} = (-1)^{\bar{F}_{ij}\bar{F}_{st}} F^{(M)}_{s,t} F^{(N)}_{i,j},$$

for $i < s < t < j$ or $s < t < i < j$,

$$(f4) \quad F^{(N)}_{t,a} F^{(M)}_{t,b} = [(-1)^{\bar{F}_{t,a}} v_t]^{NM} F^{(M)}_{t,b} F^{(N)}_{t,a}, \quad t < a < b,$$

$$(f5) \quad F^{(N)}_{b,t} F^{(M)}_{a,t} = [(-1)^{\bar{F}_{b,t}} v_t^{-1}]^{NM} F^{(M)}_{a,t} F^{(N)}_{b,t}, \quad a < b < t,$$

$$(f6) \quad F^{(M)}_{c,t} F^{(N)}_{i,c} = \sum_{0 \le j \le min\{N,M\}} v_c^{(M-j)(N-j)} F^{(N-j)}_{i,c} F^{(j)}_{i,t} F^{(M-j)}_{c,t}, \quad i < c < t.$$

Let $\mathcal{V}^0$ be the $\mathcal{A}$-algebra defined by the generators (c) and the relations [14, 2.3(g1)-(g5)] with K_{α_i} and v_i in place of K_i and v respectively.

Let $\mathcal{V}$ be the $\mathcal{A}$-superalgebra defined by the homogeneous generators (a), (b) and (c) and the relations listed above together with the relations (h1)-(h6) below(compare [14, 2.3(h1)-(h6)]):

the formulas 4.3(h1)-(h2),

$$(h3) \quad E^{(M)}_{i,i+1} F^{(N)}_{j,j+1} = F^{(N)}_{j,j+1} E^{(M)}_{i,i+1} \quad \text{if} \quad i \ne j,$$

(h4) the formula 4.3(7),

$$\left.\begin{array}{ll} (h5) & K^{\epsilon}_{\alpha_i} E^{(N)}_{\alpha_j} = v_i^{\epsilon N a_{ij}} E^{(N)}_{\alpha_j} K^{\epsilon}_{\alpha_i}, \\ (h6) & K^{\epsilon}_{\alpha_i} F^{(N)}_{\alpha_j} = v_i^{\epsilon N a_{ij}} F_{\alpha_j} K^{\epsilon}_{\alpha_i}, \end{array}\right\} \quad \epsilon = \pm 1, a_{ij} \in \tilde{A},$$

where $\tilde{A}$ is the augmented Cartan matrix defined in 2.1.

Lemma 5.1. *Let $i < c < c+1$. Then the following identities hold in $\mathcal{V}^+$.*

$$(1) \quad E^{(N)}_{i,c+1} = \sum_{j=0}^{N} (-1)^j v_c^{-j} E^{(j)}_{c,c+1} E^{(N)}_{i,c} E^{(N-j)}_{c,c+1};$$

$$(2) \quad E^{(M)}_{i,c} E^{(M+N)}_{c,c+1} E^{(N)}_{i,c} = E^{(N)}_{c,c+1} E^{(M+N)}_{i,c} E^{(M)}_{c,c+1};$$

$$(3) \quad E^{(N)}_{i,c+1} = \sum_{j=0}^{N} (-1)^j v_c^{-j} E^{(N-j)}_{i,c} E^{(N)}_{c,c+1} E^{(j)}_{i,c}.$$

$$(4) \quad E^{(N)}_{c,c+1} E^{(M)}_{i,c} = \sum_{0 \le j \le min\{N,M\}} (-1)^j v_c^{j+(N-j)(M-j)} E^{(M-j)}_{i,c} E^{(j)}_{i,c+1} E^{(N-j)}_{c,c+1}.$$

Proof. (1) We use the argument for [14, 2.5(d)]. In the right-hand side we substitute $E_{i,c}^{(N)}E_{c,c+1}^{(N-j)}$ by the expression provided by Lemma 4.11(5); applying the formula [8, 0.2 (4)] or [16, 1.3.4], we get the left-hand side.

(2) Substitute $E_{i,c}^{(M)}E_{c,c+1}^{(M+N)}$ from the left-hand side and $E_{i,c}^{(M+N)}E_{c,c+1}^{(M)}$ from the righthand side by the expression provided by Lemma 4.11(5), we get equal expressions.

(3) follows immediately from (1) and (2).

(4) If $\bar{E}_{ic} = \bar{1}$ (resp. $\bar{E}_{c,c+1} = \bar{1}$), then we get $M = 1$(resp. $N = 1$) by our convention. In the righthand side of (4), we substitute $E_{ic}^{(M)}E_{c,c+1}^{(N)}$ by the expressions provided by Lemma 4.11(5), then applying Lemma 4.11(4)(resp. Lemma 4.11(3)), we get the left-hand side of (4). Assume $\bar{E}_{ic} = \bar{E}_{c,c+1} = \bar{0}$. In the righthand side of (4), we apply Lemma 4.11(3), then substitute $E_{ic}^{(M-j)}E_{c,c+1}^{(N-j)}$ by the expression provided by Lemma 4.11(5); performing cancelations with the formula [8, 0.2 (4)], we get the left-hand side of (4). ☐

Let

$$E_0^{(\underline{N})}E_1^{\underline{\delta}} =: \Pi_{(i,j)\in I_0}E_{i,j}^{(N_{i,j})}\Pi_{(i,j)\in I_1}E_{i,j}^{(\delta_{ij})},$$

$$F_1^{\underline{\delta}}F_0^{(\underline{N})} =: \Pi_{(i,j)\in \mathcal{I}_1}F_{i,j}^{(\delta_{i,j})}\Pi_{(i,j)\in \mathcal{I}_0}F_{i,j}^{(N_{ij})}$$

($\underline{N} = (N_{ij})_{(i,j)\in\mathcal{I}_0} \in \mathbb{N}^{\mathcal{I}_0}, \underline{\delta} = (\delta_{ij})_{(i,j)\in\mathcal{I}_1} \in \{0,1\}^{|\mathcal{I}_1|}$) be the product in the order defined in Sec.3.

Proposition 5.2. *(a) $\mathcal{V}^+$ is generated as an $\mathcal{A}$-superalgebra by the elements $E_{\alpha_i}^{(N)} = E_{i,i+1}^{(N)}$ $(i \in [1, m+n), N \geq 1)$.*

(b) $\mathcal{V}^+$ is generated as an $\mathcal{A}$-module by the monomials $E_0^{(\underline{N})}E_1^{\underline{\delta}}, \underline{N} \in \mathbb{N}^{|\mathcal{I}_0|}, \underline{\delta} \in \{0,1\}^{|\mathcal{I}_1|}$.

Proof. (a) is an immediate consequence of Lemma 5.1(1).

(b) $\mathcal{V}^+$ is spanned as an $\mathcal{A}$-module by homogeneous monomials $\xi_1\xi_2\cdots\xi_L$, where each ξ_k is one of the generators $E_{ij}^{(N_{ij})}, N_{ij} \geq 1$. With respect to the order for the generators, let $\xi_l = E_{i,j}^{(N)}$ be the minimal(In case the minimal element is not unique, let ξ_l be the rightmost one). Let $|N|$ denote the sum of all N_{ij} such that $E_{ij}^{(N_{ij})} \in \{\xi_1, \xi_2, \cdots, \xi_L\}$ is minimal. We claim that each monomial can be written as an $\mathcal{A}$-linear combination of monomials $\xi_1'\xi_2'\cdots\xi_K'$ such that $\xi_1' = E_{i,j}^{(N')}$ with $0 \leq N' \leq |N|$ and $\xi_l \prec \xi_j'$ for all $j \in [2, K]$. Then the finiteness of the set $\mathcal{I}_0 \cup \mathcal{I}_1$ shows that $\xi_1\xi_2\cdots\xi_L$ equals an $\mathcal{A}$-linear combination of the monomials $E_0^{(\underline{N})}E_1^{\underline{\delta}}$.

We proceed by induction on l. Assume $\xi_{l-1} = E_{s,t}^{(M)}, M \geq 1$. The claim follows

immediately from formulas Lemma 4.11(2)-(4) in each of the following cases.

$$\begin{array}{cc} (1) & s=i; \\ (2) & t=j; \\ (3) & i<s<t<j; \\ (4) & j<s. \end{array}$$

Suppose $j=s$. By repeated application of Lemma 5.1(1), $\xi_{l-1}=E_{s,t}^{(M)}$ can be written as an $\mathcal{A}$-linear combination of terms

$$E_{t-1,t}^{(q_{t-1})}\cdots E_{j+1,j+2}^{(q_{j+1})}E_{j,j+1}^{(M)}E_{j+1,j+2}^{(M-q_{j+1})}\cdots E_{t-1,t}^{(M-q_{t-1})}, q_{t-1},\cdots,q_{j+1}\geq 0.$$

Then the claim follows from Lemma 4.11(2) and Lemma 5.1(4).

It remains to check the case $i<s<j<t$. By repeated application of Lemma 5.1(1), we write $\xi_{l-1}=E_{s,t}^{(M)}$ as an $\mathcal{A}$-linear combination of terms

$$E_{t-1,t}^{q_{t-1}}\cdots E_{j,j+1}^{(q_j)}E_{s,j}^{(M)}E_{j,j+1}^{(M-q_j)}\cdots E_{t-1,t}^{(M-q_{t-1})}, q_{t-1},\cdots,q_j\geq 0.$$

Then the claim follows from Lemma 4.11(2),(4) and Lemma 5.1(4). □

Since there is a unique super-ring isomorphism $\mathcal{V}^-\longrightarrow(\mathcal{V}^+)^{opp}$ which carries $F_{ij}^{(N)}$ to $E_{ij}^{(N)}((i,j)\in\mathcal{I}_0\cup\mathcal{I}_1)$ and v to v^{-1}, we get

Proposition 5.3. *(a) $\mathcal{V}^-$ is generated as an $\mathcal{A}$-superalgebra by the elements $F_{\alpha_i}^{(N)}=F_{i,i+1}^{(N)}(i\in[1,m+n),N\geq 1)$.*

(b) $\mathcal{V}^-$ is generated as an $\mathcal{A}$-module by the monomials

$$F_1^{\underline{\delta}}F_0^{(\underline{N})}\quad(\underline{N}\in\mathbb{N}^{\mathcal{I}_0},\underline{\delta}\in\{0,1\}^{|\mathcal{I}_1|}).$$

By [14, 2.14], $\mathcal{V}^0$ is generated as an $\mathcal{A}$-module by the elements

$$K_{\alpha_1}^{\delta_1}\cdots K_{\alpha_{m+n}}^{\delta_{m+n}}\begin{bmatrix}K_{\alpha_1};0\\ t_1\end{bmatrix}\cdots\begin{bmatrix}K_{\alpha_{m+n}};0\\ t_{n+m}\end{bmatrix},$$

$\delta_i\in\{0,1\},t_i\in\mathbb{N}$.

By a similar argument as that for [14, 2.15-2.16], we obtain a surjective $\mathcal{A}$-linear map $\pi:\mathcal{V}^-\otimes_{\mathcal{A}}\mathcal{V}^0\otimes_{\mathcal{A}}\mathcal{V}^+\to\mathcal{V}$ such that, when restricted to $\mathcal{V}^-$(resp. $\mathcal{V}^0$; $\mathcal{V}^+$), π is an $\mathcal{A}$-superalgebra homomorphism stablizing the generators. In view of the proof of [14, Prop. 2.17], we get

Proposition 5.4. *(a) $\mathcal{V}$ is generated as an $\mathcal{A}$-superalgebra by the homogeneous elements*

$$E_{\alpha_i}^{(N)},F_{\alpha_i}^{(N)},K_{\alpha_i}^{\pm 1},\begin{bmatrix}K_{\alpha_{m+n}};0\\ t\end{bmatrix},i\in[1,m+n),N\geq 0,t\geq 0.$$

(b) $\mathcal{V}$ is generated as an $\mathcal{A}$-module by elements

$$F_1^{\underline{\delta}}F_0^{\underline{N}}\Pi_{i=1}^{m+n}(K_{\alpha_i}^{\delta_i}\begin{bmatrix}K_{\alpha_i};0\\ t_i\end{bmatrix})E_0^{\underline{N}'}E_1^{\underline{\delta}'},$$

$\underline{\delta},\underline{\delta}'\in\{0,1\}^{|\mathcal{I}_1|}$, $\underline{N},\underline{N}'\in\mathbb{N}^{|\mathcal{I}_0|}$,$\delta_i\in\{0,1\},t_i\geq 0$.

We now form the $\mathcal{A}'$-superalgebras $\mathcal{V}^+_{\mathcal{A}'}$, $\mathcal{V}^-_{\mathcal{A}'}$, $\mathcal{V}^0_{\mathcal{A}'}$, and $\mathcal{V}_{\mathcal{A}'}$ by applying $-\otimes_{\mathcal{A}}\mathcal{A}'$ to $\mathcal{V}^+$, $\mathcal{V}^-$, $\mathcal{V}^0$, and $\mathcal{V}$. Denote E_{ij} and F_{ij} for $E^{(1)}_{ij}\otimes 1$ and $F^{(1)}_{ij}\otimes 1$.

Proposition 5.5. *$\mathcal{V}_{\mathcal{A}'}$ is the $\mathcal{A}'$-superalgebra defined by the generators $E_{ij}, F_{ij}((i,j)\in \mathcal{I}_1\cup\mathcal{I}_1)$, and $K^{\pm 1}_{\alpha_i}(i\in[1,m+n])$, and the following relations:*

$$(a1)\quad E^2_{ij}=0,(i,j)\in\mathcal{I}_1,$$

$$(a2)\quad E_{ij}E_{st}=(-1)^{\bar{E}_{ij}\bar{E}_{st}}E_{st}E_{ij}\quad if\quad i<s<t<j\quad or\quad s<t<i<j,$$

$$(a3)\quad E_{ta}E_{tb}=(-1)^{\bar{E}_{ta}}v_tE_{tb}E_{ta},\quad t<a<b,$$

$$(a4)\quad E_{bt}E_{at}=(-1)^{\bar{E}_{bt}}v_t^{-1}E_{at}E_{bt},\quad a<b<t,$$

$$(a5)\quad E_{ij}=E_{ic}E_{cj}-v_c^{-1}E_{cj}E_{ic},\quad i<c<j,$$

$$(b1)\quad F^2_{ij}=0,\quad (i,j)\in\mathcal{I}_1,$$

$$(b2)\quad F_{ij}F_{st}=(-1)^{\bar{F}_{ij}\bar{F}_{st}}F_{st}F_{ij}\quad if\quad i<s<t<j\quad or\quad s<t<i<j,$$

$$(b3)\quad F_{ta}F_{tb}=(-1)^{\bar{F}_{ta}}v_tF_{tb}F_{ta},\quad t<a<b,$$

$$(b4)\quad F_{bt}F_{at}=(-1)^{\bar{F}_{bt}}v_t^{-1}F_{at}F_{bt},\quad a<b<t,$$

$$(b5)\quad F_{ij}=-v_cF_{ic}F_{cj}+F_{cj}F_{ic},\quad i<c<j,$$

$$(c1)\quad K_{\alpha_i}K_{\alpha_j}=K_{\alpha_j}K_{\alpha_i},$$

$$(c2)\quad K_{\alpha_i}K^{-1}_{\alpha_i}=1,$$

$$(d1)\quad E_{\alpha_i}F_{\alpha_i}-(-1)^{\delta_{im}}F_{\alpha_i}E_{\alpha_i}=\delta_{ij}\frac{K_{\alpha_i}-K^{-1}_{\alpha_i}}{v_i-v_i^{-1}},\quad i\in[1,m+n),$$

$$\left.\begin{array}{ll}(d2)\quad K_{\alpha_i}E_{\alpha_j}=v_i^{a_{ij}}E_{\alpha_j}K_{\alpha_i}\\ (d3)\quad K_{\alpha_i}F_{\alpha_j}=v_i^{-a_{ij}}F_{\alpha_j}K_{\alpha_i}\end{array}\right\},a_{ij}\in\tilde{A}.$$

The proof is similar to that of [14, 2.19]. We need only point out that, to prove the relation Lemma 4.11(5), one can use (a5) and proceed by induction first on N with $M=1$ and then on M. By a similar argument we also get :

$\mathcal{V}^+_{\mathcal{A}'}$(resp. $\mathcal{V}^-_{\mathcal{A}'}$; $\mathcal{V}^0_{\mathcal{A}'}$) is the $\mathcal{A}'$-superalgebra defined by the generators $E_{ij}((i,j)\in\mathcal{I}_0\cup\mathcal{I}_1)$ (resp. $F_{ij}((i,j)\in\mathcal{I}_0\cup\mathcal{I}_1)$; $K^{\pm1}_{\alpha_i}(i\in[1,m+n])$ and the relations (a1)-(a5)(resp. (b1)-(b5); (c1)-(c2)).

It then follows from Coro. 4.7(V) that there is an $\mathcal{A}'$-superalgebra isomorphism $f:\mathcal{V}^0_{\mathcal{A}}\to U^0_{\mathcal{A}'}$ such that

$$f(K_{\alpha_i})=\begin{cases}K_iK^{-1}_{i+1}, & i\in[1,m+n)\\ K_{m+n}, & i=m+n.\end{cases}$$

By Th. 4.14 and [14, 2.21, 2.22], $U_{\mathcal{A}'}$ has the following $\mathcal{A}'$-basis

$$F_1^{\underline{\delta}}F_0^{\underline{N}}\Pi_{i=1}^{m+n}(K^{\delta_i}_{\alpha_i}\begin{bmatrix}K_{\alpha_i};0\\ t_i\end{bmatrix})E_0^{\underline{N}'}E_1^{\underline{\delta}'}$$

$$(\underline{\delta}, \underline{\delta}' \in \{0,1\}^{|\mathcal{I}_1|}, \delta_i \in \{0,1\}, \underline{N}, \underline{N}' \in \mathbb{N}^{|\mathcal{I}_0|}).$$

Since the relations in Prop. 5.5 are also satisfied by the generators of $U_{\mathcal{A}'}$ of the same notion(see Sec.2, Sec.3), we get a unique $\mathcal{A}'$-superalgebra homomorphism $\rho : \mathcal{V}_{\mathcal{A}'} \to U_{\mathcal{A}'}$ such that

$$\rho(E_{ij}) = E_{ij}, \rho(F_{ij}) = F_{ij}, \rho(K_{\alpha_i}) = K_{\alpha_i}.$$

By the definition of U, we obtain $\rho(\mathcal{V}) = U$(resp. $\rho(\mathcal{V}^{\pm}) = U^{\pm}$; $\rho(\mathcal{V}^0) = U^0$). By Prop. 5.4(b), ρ carries an $\mathcal{A}$-basis of $\mathcal{V}$(hence an $\mathcal{A}'$-basis of $\mathcal{V}_{\mathcal{A}'}$) to the PBW-type basis of $U_{\mathcal{A}'}$ given above. It follows that ρ is an $\mathcal{A}'$-superalgebra isomorphism, and hence $\rho_{|\mathcal{V}} : \mathcal{V} \to U$ is an $\mathcal{A}$-superalgebra isomorphism. This implies that $U_{\mathcal{A}'} \cong U \otimes_{\mathcal{A}} \mathcal{A}'$.

By induction, one obtains

$$\Delta(E_{\alpha_i}^{(N)}) = \sum_{j=0}^{N} v_i^{-j(N-j)} E_{\alpha_i}^{(j)} \otimes K_{\alpha_i}^{j} E_{\alpha_i}^{(N-j)},$$

$$\Delta(F_{\alpha_i}^{(N)}) = \sum_{j=0}^{N} v_i^{j(N-j)} F_{\alpha_i}^{(j)} K_{\alpha_i}^{-j} \otimes F_{\alpha_i}^{(N-j)}.$$

Then the $\mathcal{A}$-superalgebra U inherits a unique Hopf superalgebra structure from $U_{\mathcal{A}'}$.

Let $U(\mathfrak{g}_{\bar{0}})$ be the Hopf $\mathcal{A}$-sub-superalgebra of U generated only by even generators. Then it is easy to see that $U(\mathfrak{g}_{\bar{0}}) \otimes_{\mathcal{A}} \mathbb{F}(v) = U_q(\mathfrak{g}_{\bar{0}})$. Let $\mathcal{N}_1$(resp. $\mathcal{N}_{-1}$) denote also the $\mathcal{A}$-sub-superalgebra of U generated by the odd generators E_{ij}(resp. F_{ij}), $(i,j) \in \mathcal{I}_1$. In view of the discussion following Lemma 3.3, $\mathcal{N}_1$(resp. $\mathcal{N}_{-1}$) is a free $\mathcal{A}$-module with a basis consisting of standard monomials $E^{\underline{\delta}}$(resp. $F^{\underline{\delta}}$), $\underline{\delta} \in \{0,1\}^{|\mathcal{I}_1|}$. Therefore, we have $U = \mathcal{N}_{-1} U(\mathfrak{g}_{\bar{0}}) \mathcal{N}_1$.

6 The relations with modular representations

Assume $\mathbb{F}$ is a field of characteristic $p \neq 2$. Let G be the general linear $\mathbb{F}$-supergroup $\mathrm{GL}(m,n)$. In this section we study the relations between quantum groups and modular representations of G. We draw most of the standard results and notation from [3, 9, 12, 14, 15].

6.1 Modular representations of G

Let $\overline{U}(\mathfrak{g})_{\mathbb{Q}}$ be the enveloping superalgebra of the Lie superalgebra $\mathfrak{g}$ over $\mathbb{Q}$. For the maximal torus $\mathfrak{H}$ of $\mathfrak{g}$, let $\overline{U}(\mathfrak{H})_{\mathbb{Q}} \subseteq \overline{U}(\mathfrak{g})_{\mathbb{Q}}$ be its enveloping algebra. For each $h \in \mathfrak{H}$ and each $r \in \mathbb{N}$, set

$$\binom{h}{r} = \frac{1}{r!} h(h-1)\cdots(h-r+1) \in \overline{U}(\mathfrak{H})_{\mathbb{Q}}.$$

Defined in [3], the Kostant $\mathbb{Z}$-form $\overline{U}(\mathfrak{g})_{\mathbb{Z}}$ is a $\mathbb{Z}$-sub-superalgebra of $\overline{U}(\mathfrak{g})_{\mathbb{Q}}$ generated by

$$e_{ij}^{(r)}, f_{ij}^{(r)}((i,j) \in \mathcal{I}_0 \cup \mathcal{I}_1, r \geq 0), \binom{e_{ss}}{r}(s \in [1, m+n], r \geq 0).$$

By [3, 3.1], $\overline{U}(\mathfrak{g})_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module with a basis consisting of all the monomials of the form

$$\Pi_{(i,j)\in\mathcal{I}_1} f_{ij}^{\delta'_{ij}} \Pi_{(i,j)\in\mathcal{I}_0} f_{ij}^{(a'_{ij})} \Pi_{s=1}^{m+n} \binom{e_{ss}}{r_s} \Pi_{(i,j)\in\mathcal{I}_0} e_{ij}^{(a_{ij})} \Pi_{(i,j)\in\mathcal{I}_1} e_{ij}^{\delta_{ij}},$$

$a'_{ij}, a_{ij}, r_s \geq 0$, $\delta'_{ij}, \delta_{ij} = 0, 1$.

Recall the notation $h_{\alpha_i}, i \in [1, m+n]$ in 2.1. For each $i \in [1, m+n) \setminus m$, we get $\binom{h_{\alpha_i}}{b} \in \overline{U}(\mathfrak{g})_{\mathbb{Z}}$ $(b \in \mathbb{N})$ by [7, Coro 26.2] , while [7, Lemma 26.1] implies that $\binom{h_{\alpha_m}}{b} \in \overline{U}(\mathfrak{g})_{\mathbb{Z}}$ for all $b \in \mathbb{N}$. Note that the elements $h_{\alpha_i}, i \in [1, m+n]$ being evaluated in $\mathbb{Z}$ is equivalent to the elements $e_{ii}, i \in [1, m+n]$ being evaluated in $\mathbb{Z}$. Then using [7, Lemma 26.1] once again, we get

Lemma 6.1. *$\overline{U}(\mathfrak{g})_{\mathbb{Z}}$ has a $\mathbb{Z}$-basis consisting of all the monomials*

$$\Pi_{(i,j)\in\mathcal{I}_1} f_{ij}^{\delta'_{ij}} \Pi_{(i,j)\in\mathcal{I}_0} f_{ij}^{(a'_{ij})} \Pi_{s=1}^{m+n} \binom{h_{\alpha_s}}{r_s} \Pi_{(i,j)\in\mathcal{I}_0} e_{ij}^{(a_{ij})} \Pi_{(i,j)\in\mathcal{I}_1} e_{ij}^{\delta_{ij}},$$

$a'_{ij}, a_{ij}, r_s \geq 0$, $\delta'_{ij}, \delta_{ij} \in \{0, 1\}$.

Similarly one can describe the Kostant $\mathbb{Z}$-form $\overline{U}(\mathfrak{g}_{\bar{0}})_{\mathbb{Z}}$ and its $\mathbb{Z}$-bases.

The closed subgroups G_{ev}, P of G are defined in [3] as follows. For each commutative superalgebra A, let $P(A)$(resp. $G_{ev}(A)$) be the group of all invertible $(m+n) \times (m+n)$ matrices of the same form as the one in $G(A)$ with the additional condition $Y = 0$(resp. $Y = 0, X = 0$). Then we have $\mathrm{Lie}(P) = \mathfrak{g}^+$.

The Kostant $\mathbb{Z}$-form of $\overline{U}(\mathfrak{g}^+)_{\mathbb{Z}}$ is a $\mathbb{Z}$-free $\mathbb{Z}$-module with a basis being given by the set of all monomials of the form

$$\Pi_{(i,j),(j,i)\in\mathcal{I}_0} e_{i,j}^{(a_{ij})} \Pi_{1\leq i\leq m+n} \binom{h_{\alpha_i}}{r_i} \Pi_{(i,j)\in\mathcal{I}_1} e_{ij}^{\delta_{ij}}$$

for all $a_{ij}, r_i \in \mathbb{N}$ and $\delta_i \in \{0, 1\}$, where the product is taken in any fixed order. Then by a similar argument as that for [12, Th.3.2], we can identify $\overline{U}(\mathfrak{g}^+)_{\mathbb{F}} =: \mathbb{F} \otimes_{\mathbb{Z}} U(\mathfrak{g}^+)_{\mathbb{Z}}$ with $\mathrm{Dist}(P)$. Let $\mathfrak{N}$ be the two-sided ideal of $\overline{U}(\mathfrak{g}^+)_{\mathbb{F}}$ generated by the elements $e_{ij}, (i,j) \in \mathcal{I}_1$. Then it is easy to check that $\mathfrak{N}$ is nilpotent. Hence we can identify each simple P-module as a G_{ev}-module annihilated by $\mathfrak{N}$.

In the following, we identify Λ with $\mathbb{Z}^{m+n}$ by sending each $\lambda \in \Lambda$ to

$$(\lambda(h_{\alpha_1}), \cdots, \lambda(h_{\alpha_{m+n}})) \in \mathbb{Z}^{m+n}.$$

Let M be a Dist(G)-module. For each $\underline{z} = (z_1, \ldots, z_{m+n}) \in \mathbb{Z}^{m+n}$, define the $\underline{z}$-weight space of M(compare [3, 12]) by

$$M_{\underline{z}} = \{m \in M | \binom{h_{\alpha_i}}{r} m = \binom{z_i}{r} m \quad \text{for all} \quad i = 1, \cdots, m+n, r \geq 1\}.$$

A Dist(G)-module M is called integrable if it is locally finite over Dist(G) and satisfies $M = \sum_{\underline{z} \in \mathbb{Z}^{m+n}} M_{\underline{z}}$. By [3, 12], the category of G-modules is isomorphic to the category of integrable Dist(G)-modules.

Let

$$\mathbb{Z}_{+}^{m+n} = \{(z_1, \cdots, z_{m+n}) \in \mathbb{Z}^{m+n} | z_i \geq 0 \quad \text{for all} \quad i \neq m, m+n\}.$$

Define the set of restricted weights by

$$\mathbb{Z}_{p}^{m+n} =: \{\underline{z} \in \mathbb{Z}_{+}^{m+n} | 0 \leq z_i < p \quad \text{for all} \quad 1 \leq i \leq m+n\}.$$

Let $L(\underline{z})$(resp. $L_0(\underline{z})$) denote the simple G-module(resp. G_{ev}-module) with highest weight $\underline{z}$. Regard $L_0(\underline{z})$ as a P-module by letting $\mathfrak{N}$ act trivially. There is an induced G-module (see [3, p. 11]) defined by

$$\mathrm{Ind}_P^G \underline{z} =: \mathrm{Dist} G \otimes_{\mathrm{Dist} P} L_0(\underline{z}).$$

Proposition 6.2. *Assume $\mathbb{F}$ is algebraically closed. If $\underline{z} \in \mathbb{Z}_{+}^{m+n}$ is p-typical, then*

$$Ind_P^G \underline{z} \cong L(\underline{z}).$$

Proof. We split the proof into two cases according to whether $\underline{z} \in \mathbb{Z}_{p}^{m+n}$ or not.

Case 1. $\underline{z} \in \mathbb{Z}_{p}^{m+n}$. Since Lie$(G_{ev}) = \mathfrak{g}_{\bar{0}}$, $L_0(\underline{z})$ is a restricted simple $\mathfrak{g}_{\bar{0}}$-module. Then we obtain an isomorphism of restricted $\mathfrak{g}$-modules:

$$\mathrm{Ind}_P^G \underline{z} \cong \mathcal{K}(\underline{z}).$$

The simplicity of the $\mathfrak{g}_{\bar{0}}$-module $L_0(\underline{z})$ implies that any nonzero submodule $\mathcal{L}$ of $\mathcal{K}(\underline{z})$ contains the element $\Pi_{(i,j) \in \mathcal{I}_1} f_{ij} v^{+}$, where v^{+} is the unique maximal vector of $L_0(\underline{z})$. Then $\mathcal{L}$ contains also the element $\Pi_{(i,j) \in \mathcal{I}_1} e_{ij} \Pi_{(i,j) \in \mathcal{I}_1} f_{ij} v^{+}$. Recall the polynomial $P(\underline{z})$ in 2.1. In view of the proof of [10, Prop.2.9], we have that $\mathcal{L}$ contains $P(\underline{z}) v^{+} \in \mathbb{F}_p v^{+}$. Since $\underline{z}$ is p-typical, so that $P(\underline{z}) \neq 0$, we get $v^{+} \in \mathcal{L}$. It follows that $\mathcal{L} = \mathcal{K}(\underline{z})$. Thus, $\mathrm{Ind}_P^G \underline{z}$ is a simple restricted $\mathfrak{g}$-module. Then [12, 4.3] shows that $\mathrm{Ind}_P^G \underline{z} \cong L(\underline{z})$.

Case 2. $\underline{z} \notin \mathbb{Z}_{p}^{m+n}$. Note that $\underline{z}$ can be written uniquely as $\underline{z} = \underline{z}_1 + p\underline{z}_2$ with $\underline{z}_1 \in \mathbb{Z}_{p}^{m+n}$ being p-typical and $\underline{z}_2 \in \mathbb{Z}_{+}^{m+n}$. By the Steinberg tensor product theorem, we have an isomorphism of P-modules

$$L_0(\underline{z}) \cong L_0(\underline{z}_1) \otimes L_0(\underline{z}_2)^{[1]}.$$

By [12, Th.4.4], The simple G-module $L(\underline{z})$ is isomorphic to $L(\underline{z}_1) \otimes L_0(\underline{z}_2)^{[1]}$. Since $\underline{z}_1$ is p-typical, we get $L(\underline{z}_1) \cong \mathrm{Ind}_P^G L_0(\underline{z}_1)$ from the preceding paragraph. Clearly, the imbedding of the P-module $L_0(\underline{z}_1) \otimes L_0(\underline{z}_2)^{[1]}$ into $L(\underline{z}_1) \otimes L_0(\underline{z}_2)^{[1]}$ induces a nontrivial G-module homomorphism f from $\mathrm{Ind}_P^G \underline{z}$ into $L(\underline{z}_1) \otimes L_0(\underline{z}_2)^{[1]}$. Then the simplicity of the latter implies that f is surjective and hence isomorphic, so we get $\mathrm{Ind}_P^G \underline{z} \cong L(\underline{z})$. □

Let $\overline{U}(\mathfrak{g})_{\mathbb{F}_p} = \overline{U}(\mathfrak{g})_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_p$ and let $\overline{\mathfrak{u}}$ be the sub-superring of $\overline{U}(\mathfrak{g})_{\mathbb{F}_p}$ generated by the elements $e_{ij}, f_{ij}((i,j) \in \mathcal{I}_0 \cup \mathcal{I}_1)$ and $e_{m+n,m+n}$. For any field $\mathbb{K}$ of characteristic p, set

$$\overline{\mathfrak{u}}_{\mathbb{K}} =: \overline{\mathfrak{u}} \otimes_{\mathbb{F}_p} \mathbb{K}.$$

With the pth-power map, $\overline{\mathfrak{u}}_{\mathbb{K}}$ becomes the reduced enveloping algebra of the Lie superalgebra $\mathfrak{g} = gl(m,n)$ over $\mathbb{K}$. Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a simple $\overline{\mathfrak{u}}_{\mathbb{K}}$-module. Since the minimal polynomial for each e_{ii}(resp. h_{α_i}), $i \in [1, m+n]$ is a product of distinct linear factors, e_{ii}(resp. h_{a_i}) is diagonalizable on M. One can use similar arguments as those in [4] to get

Proposition 6.3. *Every simple $\overline{\mathfrak{u}}_{\mathbb{K}}$-module contains a unique(up to scalar multiple) homogeneous element $v^+ \neq 0$ such that $e_{ij}v^+ = 0$ for any $(i,j) \in \mathcal{I}_0 \cup \mathcal{I}_1$. There exist λ_i, $0 \leq \lambda_i \leq p-1$, such that $h_{\alpha_i} v^+ = \lambda_i v^+$ $(i \in [1, m+n])$. Non-isomorphic modules yield distinct weights $\underline{\lambda} =: (\lambda_1, \cdots, \lambda_{m+n})$. Thus there are totally p^{m+n} isomorphism classes of simple $\overline{\mathfrak{u}}_{\mathbb{K}}$-modules.*

The element v^+ in the lemma is called a maximal vector of weight $\underline{\lambda}$.

Definition 6.4. *[5, 29.13] Let $\mathfrak{A}$(rsep. $\mathfrak{A} = \mathfrak{A}_{\bar{0}} \oplus \mathfrak{A}_{\bar{1}}$) be a $\mathbb{F}$-algebra(resp. $\mathbb{F}$-superalgebra) and M a simple $\mathfrak{A}$-module. M is called absolutely simple if $M \otimes_{\mathbb{F}} \mathbb{L}$ is also a simple $\mathfrak{A} \otimes_{\mathbb{F}} \mathbb{L}$-module for any extension field $\mathbb{L} \supseteq \mathbb{F}$.*

Let $M(\underline{\lambda}) = M_{\bar{0}} \oplus M_{\bar{1}}$ be a simple $\overline{\mathfrak{u}}_{\mathbb{F}}$-module containing the unique maximal vector v^+ of weight $\underline{\lambda}$. Let $|\overline{\mathfrak{u}}_{\mathbb{F}}|$ denote the associate $\mathbb{F}$-algebra $\overline{\mathfrak{u}}_{\mathbb{F}}$ forgetting its $\mathbb{Z}_2$-structure. Then the uniqueness of the maximal vector v^+ implies that $M(\underline{\lambda})$ contains the unique maximal vector v^+ even as a $|\overline{\mathfrak{u}}_{\mathbb{F}}|$-module. Thus, each $|\overline{\mathfrak{u}}_{\mathbb{F}}|$-homomorphism from $M(\underline{\lambda})$ into itself carries v^+ to cv^+ for some $0 \neq c \in \mathbb{F}$, so we get

$$\mathrm{Hom}_{|\overline{\mathfrak{u}}_{\mathbb{F}}|}(M(\underline{\lambda}), M(\underline{\lambda})) = \mathbb{F}.$$

By [5, 29.13], $M(\underline{\lambda})$ is a absolutely simple $|\overline{\mathfrak{u}}_{\mathbb{F}}|$-module. Therefore $M(\underline{\lambda}) \otimes_{\mathbb{F}} \mathbb{L}$ is a simple $|\overline{\mathfrak{u}}_{\mathbb{L}}|$-module for any extension field $\mathbb{L}$. This implies that $M(\underline{\lambda}) \otimes_{\mathbb{F}} \mathbb{L}$ is also simple as a $\overline{\mathfrak{u}}_{\mathbb{L}}$-module, so that $M(\underline{\lambda})$ is absolute simple.

Now let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$, and let $M(\underline{\lambda})_{\overline{\mathbb{F}}}$ be a simple $\mathfrak{u}_{\overline{\mathbb{F}}}$-module possessing a unique (up to scalar multiple) maximal vector v^+ of weight $\underline{\lambda}$. Then we may identify $M(\underline{\lambda})$ with the $\mathfrak{u}_{\mathbb{F}}$-lattice $\mathfrak{u}_{\mathbb{F}} \cdot v^+ \subseteq M(\underline{\lambda})_{\overline{\mathbb{F}}}$.

Thus, the representation theory of $\mathbb{F}_p$-superalgebra $\overline{\mathfrak{u}}$ is completely determined by that of $\mathfrak{u}_{\overline{\mathbb{F}}}$. Let $\underline{\lambda}$ be as above. By [12, Lemma 4.3], $M(\underline{\lambda})_{\overline{\mathbb{F}}}$ is isomorphic to $L(\underline{\lambda})$ restricted to $\overline{\mathfrak{u}}_{\overline{\mathbb{F}}}$.

6.2 Lusztig's finite dimensional Hopf superalgebras

We fix an integer $l' \geq 1$. Let $\mathcal{B}$ be the quotient ring of $\mathcal{A}$ by the ideal generated by the $l'th$ cyclotomic polynomial $\phi_{l'} \in \mathbb{Z}[v]$. Let $l \geq 1$ be defined by

$$l = \begin{cases} l' & \text{if } l' \text{ is odd,} \\ \frac{l'}{2} & \text{if } l' \text{ is even.} \end{cases}$$

We define the $\mathcal{B}$-superalgebras $U_{\mathcal{B}}^+$, $U_{\mathcal{B}}^-$, $U_{\mathcal{B}}^0$, and $U_{\mathcal{B}}$ by applying $- \otimes_{\mathcal{A}} \mathcal{B}$ to $\mathcal{A}$-superalgebras U^+, U^-, U^0, and U. Recall the convention for the notation $E_{ij}^{(N)}$, $N \geq 0$. Let $\mathfrak{u}^+$, $\mathfrak{u}^-$, $\mathfrak{u}^0$, and $\mathfrak{u}$ be the $\mathcal{B}$-sub-superalgebras of $U_{\mathcal{B}}^+$, $U_{\mathcal{B}}^-$, $U_{\mathcal{B}}^0$, and $U_{\mathcal{B}}$ generated respectively by the elements

$$E_{ij}^{(N)}((i,j) \in \mathcal{I}_0 \cup \mathcal{I}_1, N \in [0,l) \quad); F_{ij}^{(N)}((i,j) \in \mathcal{I}_0 \cup \mathcal{I}_1, N \in [0,l) \quad);$$

$$K_{\alpha_s}^{\pm 1}, \begin{bmatrix} K_{\alpha_s}; 0 \\ t \end{bmatrix} (s \in [1, m+n], t \in [0,l) \quad);$$

and

$$E_{ij}^{(N)}, F_{ij}^{(N)}, \begin{bmatrix} K_{\alpha_{m+n}}; 0 \\ t \end{bmatrix}, K_{\alpha_s}^{\pm 1}((i,j) \in \mathcal{I}_0 \cup \mathcal{I}_1, s \in [1, m+n], t, N \in [0,l) \quad).$$

Using a similar argument as that in [14, 5.2, 5.3], one can get a description of $\mathfrak{u}$ and its sub-superalgebras in terms of generators and relations. Moreover, we get

Proposition 6.5. *(a) $\mathfrak{u}^+$ is generated as a $\mathcal{B}$-superalgebra by the elements $E_{\alpha_i}^{(N)}(i \in [1, m+n), N \in [0,l))$ and as a $\mathcal{B}$-module by the elements $E_0^{(\underline{N})} E_1^{\underline{\delta}}$, $\underline{N} \in [0,l)^{|\mathcal{I}_0|}$, $\underline{\delta} \in \{0,1\}^{|\mathcal{I}_1|}$.*

(b) $\mathfrak{u}^-$ is generated as a $\mathcal{B}$-superalgebra by the elements $F_{\alpha_i}^{(N)}(i \in [1, m+n), N \in [0,l))$ and as a $\mathcal{B}$-module by the elements $F_1^{\underline{\delta}} F_0^{(\underline{N})}$, $\underline{N} \in [0,l)^{|\mathcal{I}_0|}, \underline{\delta} \in \{0,1\}^{|\mathcal{I}_1|}$.

(c) $\mathfrak{u}^0$ is generated as a $\mathcal{B}$-module by the basis elements $\Pi_{i=1}^{m+n}(K_{\alpha_i}^{\delta_i} \begin{bmatrix} K_{\alpha_i}; 0 \\ t_i \end{bmatrix})$, $t_i \in [0,l)$, $\delta_i \in \{0,1\}$.

(d) $\mathfrak{u}$ is generated as a $\mathcal{B}$-superalgebra by the elements $E_{\alpha_i}^{(N)}, F_{\alpha_i}^{(N)}$ $(i \in [1, m+n)$, $N \in [0,l))$ and $K_{\alpha_j}^{\pm 1}(j \in [1, m+n]$ $)$ and as a $\mathcal{B}$-module by the elements

$$F_1^{\underline{\delta}} F_0^{(\underline{N})} \Pi_{i=1}^{m+n}(K_{\alpha_i}^{\delta_i} \begin{bmatrix} K_{\alpha_i}; 0 \\ t_i \end{bmatrix}) E_0^{(\underline{N'})} E_1^{\underline{\delta'}}$$

$\underline{N}, \underline{N'} \in [0,l)^{|\mathcal{I}_0|}, \underline{\delta}, \underline{\delta'} \in \{0,1\}^{|\mathcal{I}_1|}$, $\delta_i \in \{0,1\}, t_i \in [0,l)$.

(e) $\mathfrak{u}^+$, $\mathfrak{u}^-$, $\mathfrak{u}^0$ and $\mathfrak{u}$ are free $\mathcal{B}$-superalgebras of rank

$$2^{mn} l^{\binom{m}{2}+\binom{n}{2}}, \quad 2^{mn} l^{\binom{m}{2}+\binom{n}{2}}, \quad (2l)^{m+n}, \quad \text{and} \quad 2^{mn+1} l^{\binom{m}{2}+\binom{n}{2}} + (2l)^{mn}$$

respectively.

Let $\mathcal{B}'$ be the quotient field of $\mathcal{B}$. We form the $\mathcal{B}'$-superalgebras $'\mathfrak{u}^+$, $'\mathfrak{u}^-$, $'\mathfrak{u}^0$, $'\mathfrak{u}$ and $U_{\mathcal{B}'}$ by applying $-\otimes_{\mathcal{B}}\mathcal{B}'$ to the $\mathcal{B}$-algebras $\mathfrak{u}^+$, $\mathfrak{u}^-$, $\mathfrak{u}^0$, $\mathfrak{u}$, and $U_{\mathcal{B}}$ respectively. Then Lusztig's argument [14, 5.7,5.8] can be applied almost verbatim to obtain the following results:

(1) $'\mathfrak{u}^+$ is defined by the generators $E_{ij}((i,j)\in\mathcal{I}_0\cup\mathcal{I}_1)$ and the relations (a1)-(a5) in Prop. 5.5 and $E_{ij}^l=0((i,j)\in\mathcal{I}_0)$.

(2) $'\mathfrak{u}^-$ is defined by the generators $F_{ij}((i,j)\in\mathcal{I}_0\cup\mathcal{I}_1)$ and the relations (b1)-(b5) in Prop. 5.5 and $F_{ij}^l=0((i,j)\in\mathcal{I}_0)$.

(3) $'\mathfrak{u}^0$ is defined by the generators $K_{\alpha_i}(i\in[1,m+n])$ and the relations (c1), (c2) in Prop. 5.5 and $K_{\alpha_i}^{2l}=1$.

(4) $'\mathfrak{u}$ is defined by the generators E_{ij}, $F_{ij}((i,j)\in\mathcal{I}_0\cup\mathcal{I}_1)$, and $K_{\alpha_i}(i\in[1,m+n])$ and the relations (a1)–(d3) in Prop. 5.5 and $E_{ij}^l=0, F_{ij}^l=0$, $K_{\alpha_i}^{2l}=1$.

(5) $'\mathfrak{u}^+, {}'\mathfrak{u}^-, {}'\mathfrak{u}^0, {}'\mathfrak{u}$ may be regarded as $\mathcal{B}'$-sub-superalgebras of $U_{\mathcal{B}'}$ having the following bases:

$$'\mathfrak{u}^+ : \Pi_{(i,j)\in\mathcal{I}_0}E_{i,j}^{N_{ij}}\Pi_{(i,j)\in\mathcal{I}_1}E_{i,j}^{\delta_{ij}},\quad N_{ij}\in[0,l),\delta_{ij}\in\{0,1\};$$

$$'\mathfrak{u}^- : \Pi_{(i,j)\in\mathcal{I}_1}F_{i,j}^{\delta'_{i,j}}\Pi_{(i,j)\in\mathcal{I}_0}F_{i,j}^{N'_{ij}},\quad N'_{ij}\in[0,l),\delta'_{ij}\in\{0,1\};$$

$$'\mathfrak{u}^0 : \Pi_{i=1}^{m+n}K_{\alpha_i}^{N_i},\quad N_i\in[0,2l);$$

$$'\mathfrak{u} : \Pi_{(i,j)\in\mathcal{I}_1}F_{i,j}^{\delta'_{ij}}\Pi_{(i,j)\in\mathcal{I}_0}F_{i,j}^{N'_{ij}}\Pi_{i=1}^{m+n}K_{\alpha_i}^{N_i}\Pi_{(i,j)\in\mathcal{I}_0}E_{i,j}^{N_{ij}}\Pi_{(i,j)\in\mathcal{I}_1}E_{i,j}^{\delta_{ij}},$$

$N_{ij}, N'_{ij}\in[0,l)$, $N_i\in[0,2l)$, $\delta_{ij},\delta'_{ij}\in\{0,1\}$.

In the following we assume $l=l'$ is odd. Use Lusztig's notion

$$K_{i,t}=K_{\alpha_i}^{-t}\begin{bmatrix}K_{\alpha_i};0\\ t\end{bmatrix}, 1\le i\le m+n, t\ge 0.$$

Then by [14, Lemma 6.4], the elements $\Pi_{i=1}^{m+n}K_{\alpha_i}^{l\delta_i}\Pi_{i=1}^{m+n}K_{i,t_i}$ $(0\le t_i,\ \delta_i\in\{0,1\})$ form a $\mathcal{B}$-basis of $U_{\mathcal{B}}^0$.

Following [14], we let $\tilde{U}_{\mathcal{B}}$, $\tilde{\mathfrak{u}}$(resp. $\tilde{U}_{\mathcal{B}'}$, $'\tilde{\mathfrak{u}}$)be the quotient of $\mathcal{B}$-superalgebras(resp. $\mathcal{B}'$-superalgebras) of $U_{\mathcal{B}}$, $\mathfrak{u}$(resp. $U_{\mathcal{B}'}$, $'\mathfrak{u}$) by the two-sided ideal generated by the central elements

$$K_{\alpha_1}^l-1,\cdots,K_{\alpha_{m+n}}^l-1.$$

Then we get:

(a) The elements

$$F_1^{\underline{\delta}}F_0^{(\underline{N})}\Pi_{i=1}^{m+n}K_{i,t_i}E_0^{(\underline{N}')}E_1^{\underline{\delta}'}\quad(\underline{N},\underline{N}'\in\mathbb{N}^{|\mathcal{I}_0|},\underline{\delta},\underline{\delta}'\in\{0,1\}^{|\mathcal{I}_1|},t_i\in\mathbb{N})$$

form a $\mathcal{B}$-basis of $\tilde{U}_{\mathcal{B}}$ and $\mathcal{B}'$-basis of $\tilde{U}_{\mathcal{B}'}$.

(b) The elements

$$F_1^{\underline{\delta}}F_0^{(\underline{N})}\Pi_{i=1}^{m+n}K_{i,t_i}E_0^{(\underline{N}')}E_1^{\underline{\delta}'}\quad(\underline{N},\underline{N}'\in[0,l)^{|\mathcal{I}_0|},\underline{\delta},\underline{\delta}'\in\{0,1\}^{|\mathcal{I}_1|},t_i\in[0,l))$$

form a $\mathcal{B}$-basis of $\tilde{\mathfrak{u}}$ and $\mathcal{B}'$-basis of $'\tilde{\mathfrak{u}}$.

Let k be a commutative ring and let q be an invertible element in k. Set

$$U_{q,k} = U \otimes_{\mathcal{A}} k, \quad U(\mathfrak{g}_{\bar{0}})_{q,k} = U(\mathfrak{g}_{\bar{0}}) \otimes_{\mathcal{A}} k,$$

where k is regarded as an $\mathcal{A}$-algebra with v acting as the multiplication by q. For brevity we denote

$$U_v =: U_{v,\mathbb{C}(v)}, \quad U(\mathfrak{g}_{\bar{0}})_v =: U(\mathfrak{g}_{\bar{0}})_{v,\mathbb{C}(v)}, \quad U_\eta =: U_{\eta,\mathbb{C}}, \quad U(\mathfrak{g}_{\bar{0}})_\eta =: U(\mathfrak{g}_{\bar{0}})_{\eta,\mathbb{C}},$$

$$U_{\mathbb{Z}} = U_{1,\mathbb{Z}}, \quad U(\mathfrak{g}_{\bar{0}})_{\mathbb{Z}} = U(\mathfrak{g}_{\bar{0}})_{1,\mathbb{Z}}, \quad U_{\mathbb{Q}} = U_{1,\mathbb{Q}},$$

where η is a primitive lth root of unity. For the $\mathcal{A}$-sub-superalgebras $\mathcal{N}_{\pm 1}$, $\mathcal{N}^+_{\pm 1}$ of U, similar notation are defined. But we will usually omit the subscripts v and η and write just $\mathcal{N}_{\pm 1}$ and $\mathcal{N}^+_{\pm 1}$. We denote by $\tilde{U}_{q,k}$ the quotient superring of $U_{q,k}$ by its two-sided ideal generated by the central elements $K^l_{\alpha_i} - 1, i \in [1, m+n]$.

Repeating the arguments used to prove [14, 6.7(a),(c)], one obtains easily the following result.

Proposition 6.6. *There is an isomorphism of superalgebras* $\phi\colon \overline{U}(\mathfrak{g})_{\mathbb{Q}} \longrightarrow \tilde{U}_{\mathbb{Q}}$ *such that*

$$\phi(e_{ij}) = E_{ij}, \phi(f_{ij}) = F_{ij}, (i,j) \in \mathcal{I}_0 \cup \mathcal{I}_1, \phi(h_{\alpha_s}) = \begin{bmatrix} K_{\alpha_s}; 0 \\ 1 \end{bmatrix}, s \in [1, m+n].$$

In particular, we have $\phi(\overline{U}(\mathfrak{g})_{\mathbb{Z}}) = \tilde{U}_{\mathbb{Z}}$.

6.3 Representations of the Hopf superalgebra $'\tilde{\mathfrak{u}}$

Let $Y = Y_{\bar{0}} \oplus Y_{\bar{1}}$ be a U_v-module. For $\underline{z} = (z_1, \ldots, z_{m+n}) \in \mathbb{Z}^{m+n}$, we define its weight space by

$$Y_{\underline{z}} = \{y \in Y | K_{\alpha_i} y = v_i^{z_i} y, i = 1, \ldots, m+n\}.$$

Note that U_v is a special case of U_q (see Sec.2) with $\mathbb{F} = \mathbb{C}$. Then by similar discussions as those in Sec. 4.2, the sum $\sum_{\underline{z}} Y_{\underline{z}}$ is direct and $\mathbb{Z}_2$-graded.

Assume $\underline{z} = (z_1, \ldots, z_{m+n}) \in \mathbb{Z}^{m+n}$. We define a U_v-module structure on U_v^- by letting F_{α_i} act as left multiplication, $E_{\alpha_i} 1 = 0$, $K_{\alpha_j} 1 = v_i^{z_j} 1$, $i \in [1, m+n)$, $j \in [1, m+n]$. We denote this U_v-module by $Y(\underline{z})$.

Applying a similar argument as that in [15, 6.1], we see that $Y(\underline{z})$ has a unique simple quotient $\overline{Y}(\underline{z})$. Both $Y(\underline{z})$ and $\overline{Y}(\underline{z})$ are $\mathbb{Z}_2$-graded and are direct sums of their weight spaces which are finite dimensional $\mathbb{C}(v)$-spaces.

Let $\underline{z} \in \mathbb{Z}_+^{m+n}$, and let $\mathcal{L}_0(\underline{z})$ be a simple $U(\mathfrak{g}_{\bar{0}})_v$-module of highest weight $\underline{z}$. Then $\mathcal{L}_0(\underline{z})$ is finite dimensional. Recall that U_v has as the subalgebra $U(\mathfrak{g}_{\bar{0}})_v \mathcal{N}_1$.

Regard $\mathcal{L}_0(\underline{z})$ as a $U(\mathfrak{g}_{\bar{0}})_v\mathcal{N}_1$-module by letting $U(\mathfrak{g}_{\bar{0}})_v\mathcal{N}_1^+$ act trivially. Then the induced U_v-module

$$\mathcal{K}(\underline{z}) = U_v \otimes_{U(\mathfrak{g}_{\bar{0}})_v\mathcal{N}_1} \mathcal{L}_0(\underline{z})$$

is also finite dimensional.

Lemma 6.7. *For every $\underline{z} \in \mathbb{Z}_+^{m+n}$, $\overline{Y}(\underline{z})$ is a homomorphic image of $\mathcal{K}(\underline{z})$.*

Proof. Let $1 \in \overline{Y}(\underline{z})$ be the unique maximal vector. We claim that $U(\mathfrak{g}_{\bar{0}})_v\mathcal{N}_1 \cdot 1 \subseteq \overline{Y}(\underline{z})$ is a simple $U(\mathfrak{g}_{\bar{0}})_v\mathcal{N}_1$-submodule. If not, then it contains a proper simple $U(\mathfrak{g}_{\bar{0}})_v\mathcal{N}_1$-submodule M'. From the discussion following Lemma 4.12, M' has to be a simple $U(\mathfrak{g}_{\bar{0}})_v$-module annihilated by $U(\mathfrak{g}_{\bar{0}})_v\mathcal{N}_1^+$, so that M' contains a unique maximal vector v^+ which by definition is a nonzero element with $E_{\alpha_i}v^+ = 0$ for all $i \in [1, m+n) \setminus m$. Since v^+ is annihilated by $E_{\alpha_m} \in U(\mathfrak{g}_{\bar{0}})_v\mathcal{N}_1^+$, v^+ is also a maximal vector for the U_v-module $\overline{Y}(\underline{z})$, i.e., $v^+ = c \cdot 1$ for some $c \in \mathbb{C}(v)$. Hence we get $M' = \overline{Y}(\underline{z})$, a contradiction.

Thus, we have $\mathcal{L}_0(\underline{z}) \cong U(\mathfrak{g}_{\bar{0}})_v\mathcal{N}_1 \cdot 1 \subseteq \overline{Y}(\underline{z})$. This inclusion induces a U_v-homomorphism from $\mathcal{K}(\underline{z})$ into $\overline{Y}(\underline{z})$ that has to be surjective because $\overline{Y}(\underline{z})$ is simple. $\square$

Assume $l = l'$ is an odd integer ≥ 3. In what follows, we identify $\mathcal{B}'$ with the subfield $\mathbb{Q}(\eta)$ of $\mathbb{C}$ by identifying v with η. Then we may regard $U_{\mathcal{B}'}$ as a Hopf $\mathcal{B}'$-sub-superalgebra of U_η such that $U_{\mathcal{B}'} \otimes_{\mathcal{B}'} \mathbb{C} \cong U_\eta$.

Set

$$\eta_i = \begin{cases} \eta, & \text{if } i \leq m \\ \eta^{-1}, & \text{if } m+1 \leq i \leq m+n. \end{cases}$$

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a U_η-module(resp. $U_{\mathcal{B}'}$-module) of type $\underline{1}$. For each $\underline{z} = (z_1, \cdots, z_{m+n}) \in \mathbb{Z}^{m+n}$, we define the $\underline{z}$-weight space

$$V_{\underline{z}} = \{x \in V | K_{\alpha_i}x = \eta_i^{z_i}x, \begin{bmatrix} K_{\alpha_i}; 0 \\ l \end{bmatrix} x = \begin{bmatrix} z_i \\ l \end{bmatrix}_{\eta_i} x \quad \text{for} \quad 1 \leq i \leq m+n\}.$$

Since the parity of K_i, $\begin{bmatrix} K_{\alpha_j}; 0 \\ l \end{bmatrix}$ and $\begin{bmatrix} K_{m+n}; 0 \\ l \end{bmatrix}$ are all $\bar{0}$, $V_{\underline{z}}$ is $\mathbb{Z}_2$-graded. Then the argument in [15, 5.2] can be applied almost verbatim to get that $\sum_{\underline{z}} V_{\underline{z}}$ is a U_η-submodule(resp. $U_{\mathcal{B}'}$-submodule) of V. Also by [15, 3.3(b)], the sum $\sum_{\underline{z}} V_{\underline{z}}$ is direct.

Definition 6.8. *A U_η-module(resp. $U_{\mathcal{B}'}$-module) $V = V_{\bar{0}} \oplus V_{\bar{1}}$ of type $\underline{1}$ is called* integral *if $V = \sum_{\underline{z} \in \mathbb{Z}^{m+n}} V_{\underline{z}}$.*

Note: Since [15, 5.1(b)] fails for $\begin{bmatrix} K_{\alpha_i}; 0 \\ l \end{bmatrix}$, $i \in \{m, m+n\}$, not every finite dimensional simple U_η-module(resp. $U_{\mathcal{B}'}$-module) is integral.

We shall now construct the integral simple modules for U_η. Let $\underline{z} \in \mathbb{Z}^{m+n}$. Denote by $Y_{\mathcal{A}}(\underline{z})$(resp. $\overline{Y}_{\mathcal{A}}(\underline{z})$) the U-invariant $\mathcal{A}$-lattice $U \cdot 1$ of $Y(\underline{z})$(resp. $\overline{Y}(\underline{z})$). Set

$$Y_\eta(\underline{z}) = Y_{\mathcal{A}}(\underline{z}) \otimes_{\mathcal{A}} \mathbb{C}, \quad \overline{Y}_\eta(\underline{z}) = \overline{Y}_{\mathcal{A}}(\underline{z}) \otimes_{\mathcal{A}} \mathbb{C},$$

where $\mathbb{C}$ is regarded as an $\mathcal{A}$-algebra by letting v act as multiplication by η. Then $Y_\eta(\underline{z})$, $\overline{Y}_\eta(\underline{z})$ are naturally U_η-modules. Let $L_\eta(\underline{z})$ be the unique simple quotient of $Y_\eta(\underline{z})$(or, equivalently, of $\overline{Y}_\eta(\underline{z})$). Then clearly $L_\eta(\underline{z})$ is integral. If $\underline{z} \in \mathbb{Z}_+^{m+n}$, Lemma 6.7 implies that $L_\eta(\underline{z})$ is finite dimensional.

Assume $\underline{z} \in \mathbb{Z}_+^{m+n}$. By Lemma 6.7, there is a U_v-epimorphism $f\colon \mathcal{K}(\underline{z}) \longrightarrow \overline{Y}(\underline{z})$. Since $\mathcal{L}_0(\underline{z})$ is a simple $U(\mathfrak{g}_{\bar{0}})_v$-submodule, $f|_{\mathcal{L}_0(\underline{z})}$ is a $U(\mathfrak{g}_{\bar{0}})_v$-isomorphism onto its image (denoted also $\mathcal{L}_0(\underline{z})$) in $\overline{Y}(\underline{z})$. Besides, $\mathcal{L}_0(\underline{z}) \subseteq \overline{Y}(\underline{z})$ is also annihilated by $U(\mathfrak{g}_{\bar{0}})_v \mathcal{N}_1^+$. Then we have $\overline{Y}(\underline{z}) = \mathcal{N}_{-1}\mathcal{L}_0(\underline{z})$, so that

$$\overline{Y}_{\mathcal{A}}(\underline{z}) = \mathcal{N}_{-1} U(\mathfrak{g}_{\bar{0}}) \cdot v^+,$$

where v^+ is the unique maximal vector in $\mathcal{L}_0(\underline{z})$. Denote the image of v^+ in $L_\eta(\underline{z})$ also by v^+. It then follows that $L_\eta(\underline{z})$ has as a $U(\mathfrak{g}_{\bar{0}})_\eta$-submodule $U(\mathfrak{g}_{\bar{0}})_\eta v^+$ annihilated by $U(\mathfrak{g}_{\bar{0}})_\eta \mathcal{N}_1^+$ and

$$L_\eta(\underline{z}) = \mathcal{N}_{-1} U(\mathfrak{g}_{\bar{0}})_\eta v^+.$$

In view of the proof for [15, 6.4], we get

Proposition 6.9. *The map $\underline{z} \longrightarrow L_\eta(\underline{z})$ defines a bijection between $\mathbb{Z}_+^{m+n}$ and the set of isomorphism classes of integral simple U_η-modules of type $\underline{1}$, of finite dimension over $\mathbb{C}$.*

Let $\mathbb{K}$ be any intermediate field between $\mathbb{Q}(\eta)$ and $\mathbb{C}$. Set $'\tilde{\mathfrak{u}}_{\mathbb{K}} =' \tilde{\mathfrak{u}} \otimes_{\mathcal{B}'} \mathbb{K}$. We now study $'\tilde{\mathfrak{u}}_{\mathbb{K}}$-modules. They will be assumed to be finite dimensional over $\mathbb{K}$. Since $K_{\alpha_i}^l = 1 (i \in [1, m+n])$ in $'\tilde{\mathfrak{u}}$, each $'\tilde{\mathfrak{u}}_{\mathbb{K}}$-module $M = M_{\bar{0}} \oplus M_{\bar{1}}$ has a decomposition

$$M_{\bar{0}} = \oplus_{\underline{h}} (M_{\bar{0}})_{\underline{h}}, \quad M_{\bar{1}} = \oplus_{\underline{h}'} (M_{\bar{1}})_{\underline{h}'},$$

where $\underline{h}, \underline{h}' \in [0, l)^{m+n}$ and

$$(M_{\bar{j}})_{\underline{h}} = \{x \in M_{\bar{j}} | K_{\alpha_i} x = \eta_i^{h_i} x, 1 \le i \le m+n\}, \bar{j} \in \mathbb{Z}_2.$$

Let

$$M^0 = \{x \in \mathcal{H}(M) | E_{ij} x = 0, \quad \text{for all} \quad (i,j) \in \mathcal{I}_0 \cup \mathcal{I}_1\}.$$

A nonzero vector $v \in M^0 \cap (M_{\bar{i}})_{\underline{h}}$ is called a maximal vector of weight $\underline{h}$. Then applying verbatim [14, 5.10,5.11] we get

Proposition 6.10. *Each simple $'\tilde{\mathfrak{u}}_{\mathbb{K}}$-module $M = M_{\bar{0}} \oplus M_{\bar{1}}$ contains a unique (up to scalar multiple) maximal vector of weight $\underline{h} \in [0, l)^{m+n}$. The correspondence $M \mapsto \underline{h}$ defines a bijection between the set of isomorphism classes of simple $'\tilde{\mathfrak{u}}_{\mathbb{K}}$-modules and the set $[0, l)^{m+n}$.*

Using the fact that each simple ${}'\tilde{\mathfrak{u}}_{\mathbb{K}}$-module M contains a unique maximal vector, together with a similar discussion as that in Sec. 6.1, we see that M is absolutely simple. So we may restrict our attention to just the case $\mathbb{K} = \mathbb{C}$. It follows from the description of the bases of superalgebras $\tilde{U}_{\mathcal{B}'}$ and ${}'\tilde{\mathfrak{u}}$ in 6.2 that ${}'\tilde{\mathfrak{u}}_{\mathbb{C}}$ can be regarded as a sub-superalgebra of $\tilde{U}_\eta$.

Set

$$\mathbb{Z}_l^{m+n} =: \{(z_1, \ldots, z_{m+n}) \in \mathbb{Z}_+^{m+n} | 0 \le z_i \le l-1 \quad \text{for all} \quad i \ne m, m+n\}.$$

The following lemma can be proved by a similar argument as that for [15, 7.1].

Lemma 6.11. *Assume $\underline{z} \in \mathbb{Z}_l^{m+n}$ and let x be a maximal vector of $L_\eta(\underline{z})$. Then*

(a) $F_{\alpha_i}^{(l)} x = 0$ for $i \in [1, m+n) \setminus m$.

(b) Let $\nabla = \{y \in \mathcal{H}(L_\eta(\underline{z})) | E_{\alpha_i} y = 0 \quad \text{for all} \quad i \ne m, m+n\}$. Then $\nabla = \mathbb{C} \cdot x$.

(c) Then restriction of $L_\eta(\underline{z})$ to ${}'\tilde{\mathfrak{u}}_{\mathbb{C}}$ is a simple ${}'\tilde{\mathfrak{u}}_{\mathbb{C}}$-module.

(d) $L_\eta(\underline{z}) =' \tilde{\mathfrak{u}}_{\mathbb{C}} \cdot x$.

It then follows that each simple ${}'\tilde{\mathfrak{u}}_{\mathbb{C}}$-module can be lifted to an integral simple U_η-module of type $\underline{1}$.

6.4 The extended Lusztig conjecture

Recall the $\mathbb{Z}$-superalgebra $\overline{U}(\mathfrak{g})_{\mathbb{Z}}$. For any field $\mathbb{F}$, we denote $\overline{U}(\mathfrak{g})_{\mathbb{F}} = \overline{U}(\mathfrak{g})_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}$, where $\mathbb{F}$ is regarded as a $\mathbb{Z}$-algebra by letting $z \in \mathbb{Z}$ act as multiplication by $z1_{\mathbb{F}} \in \mathbb{F}$. For the $\mathbb{Z}$-subalgebra $\overline{U}(\mathfrak{g}_{\bar{0}})_{\mathbb{Z}}$, the notation $\overline{U}(\mathfrak{g}_{\bar{0}})_{\mathbb{F}}$ is defined similarly.

By [3, Th.3.2], $\overline{U}(\mathfrak{g})_{\mathbb{F}}$ is isomorphic to $\mathrm{Dist}(G)$ as Hopf superalgebras if $char.\mathbb{F} \ne 2$, where $\mathrm{Dist}(G)$ is the distribution superalgebra of the $\mathbb{F}$-supergroup $G = \mathrm{GL}(m, n)$.

In this subsection assume $l = l'$ is an odd prime p, and assume $\mathbb{F}$ is an algebraically closed field of characteristic p. Consider the ring homomorphism $\mathcal{B} \longrightarrow \mathbb{F}_p$ which takes $z \in \mathbb{Z}$ to z mod $p \in \mathbb{F}_p$ and v to 1, and let $\mathbf{m}$ be its kernel. Then applying a similar argument as that for [14, Th. 6.8], we get

Theorem 6.12. *There are isomorphisms of Hopf superalgebras:*

$$\tilde{U}_{\mathcal{B}} / \mathbf{m} \tilde{U}_{\mathcal{B}} \cong \overline{U}(\mathfrak{g})_{\mathbb{F}_p}, \quad \tilde{\mathfrak{u}} / \mathbf{m} \tilde{\mathfrak{u}} \cong \overline{\mathfrak{u}}.$$

Assume η is a primitive pth root of unity. By [9, Ch.H], $\mathbb{Z}[\eta]$ is the ring of all algebraic integers in $\mathbb{Q}(\eta)$ and $1 - \eta$ generates the unique maximal ideal in $\mathbb{Z}[\eta]$. Let $\mathcal{R}$ denote the localization of $\mathbb{Z}[\eta]$ at $(1 - \eta)$. Then $\mathcal{R}$ is a discrete valuation ring with residue field $\mathbb{F}_p$. Regard the field $\mathbb{F}$ as a $\mathcal{R}$-algebra via the embedding of the residue field $\mathcal{R}$ into $\mathbb{F}$. We can identify $U_{\eta,\mathcal{R}} \otimes_{\mathcal{R}} \mathbb{F}$ with $U_{1,\mathbb{F}}$(see Sec. 6.3).

Assume $\underline{z} \in \mathbb{Z}_+^{m+n}$. Let v^+ be a maximal vector of the simple U_η-module $L_\eta(\underline{z})$. Then $L_{\mathcal{R}}(\underline{z}) = U_{\eta,\mathcal{R}} v^+$ is a $U_{\eta,\mathcal{R}}$-invariant $\mathcal{R}$-lattice in $L_\eta(\underline{z})$. Now

$$L_\eta(\underline{z})_{\mathbb{F}} = L_\eta(\underline{z})_{\mathcal{R}} \otimes_{\mathcal{R}} \mathbb{F}$$

has a natural structure as a module over $U_{1,\mathbb{F}}$. Since each K_{α_i} acts on $L_\eta(\underline{z})_{\mathbb{F}}$ as the identity, $L_\eta(\underline{z})_{\mathbb{F}}$ is a $\tilde{U}_{1,\mathbb{F}}$-module. By Prop. 6.6, we have

$$\tilde{U}_{1,\mathbb{F}} = \tilde{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F} \cong \overline{U}(\mathfrak{g})_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F} = \overline{U}(\mathfrak{g})_{\mathbb{F}},$$

so that $L_\eta(\underline{z})_{\mathbb{F}}$ is a $\overline{U}(\mathfrak{g})_{\mathbb{F}}$-module. Then [3, Th. 3.2] says that $L_\eta(\underline{z})_{\mathbb{F}}$ is a Dist(G)-module, and hence a G-module by [3, Coro. 3.5].

By the theorem above, each simple $\bar{\mathfrak{u}}$-module corresponding to a simple $'\tilde{\mathfrak{u}}$-module M and has dimension $\le dim M$. We now extend the Lusztig's conjecture in [14, 0.3] to the super case as follows.

Conjecture: If $p > max\{m, n\}$ and $\underline{z} \in \mathbb{Z}_p^{m+n}$, then the inequality above is an equality and $\bar{\mathfrak{u}}$ and $'\tilde{\mathfrak{u}}$ have identical representation theories.

The conjecture is supported by the following theorem.

Theorem 6.13. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > max\{m, n\}$. Assume Lusztig's conjecture in [14, 0.3]. If $\underline{z} \in \mathbb{Z}_p^{m+n}$ is p-typical, then $L_\eta(\underline{z})_{\mathbb{F}}$ is simple as a $GL(m, n)$-module.*

Proof. Let v^+ be a maximal vector of $L_\eta(\underline{z})$. From the discussion preceding Prop. 6.9, we have

$$L_{\mathcal{R}}(\underline{z}) = U_{\eta,\mathcal{R}} v^+ = \mathcal{N}_{-1,\mathcal{R}} U(\mathfrak{g}_{\bar{0}})_{\eta,\mathcal{R}} v^+,$$

where $U(\mathfrak{g}_{\bar{0}})_{\eta,\mathcal{R}} v^+$ is a $U(\mathfrak{g}_{\bar{0}})_{\eta,\mathcal{R}}$-invariant lattice of the simple $U(\mathfrak{g}_{\bar{0}})_\eta$-module annihilated by $U(\mathfrak{g}_{\bar{0}})_\eta \mathcal{N}_1^+$. This gives us

$$L_\eta(\underline{z})_{\mathbb{F}} = \wedge(\mathfrak{g}_{-1})_{\mathbb{F}} \overline{U}(\mathfrak{g}_{\bar{0}})_{\mathbb{F}} v^+,$$

where $\wedge(\mathfrak{g}_{-1})$ is the $\mathbb{F}$-subalgebra of $\overline{U}(\mathfrak{g})_{\mathbb{F}}$ generated by $f_{ij}, (i, j) \in \mathcal{I}_1$. Since $\underline{z} \in \mathbb{Z}_p^{m+n}$ and $p > max\{m, n\}$, the Lusztig conjecture in [14, 0.3] says that $\overline{U}(\mathfrak{g}_{\bar{0}})_{\mathbb{F}} v^+$ is a simple $\overline{U}(\mathfrak{g}_{\bar{0}})_{\mathbb{F}}$-module of highest weight $\underline{z}$. Therefore $L_\eta(\underline{z})_{\mathbb{F}}$ is a homomorphic image of $\mathcal{K}(\underline{z})$.

Since $\underline{z}$ is p-typical, by Proposition 6.2, we obtain

$$L_\eta(\underline{z})_{\mathbb{F}} \cong \mathcal{K}(\underline{z}) \cong \mathrm{Dist} G \otimes_{\mathrm{Dist} P} \overline{U}(\mathfrak{g}_{\bar{0}})_{\mathbb{F}} v^+ = Ind_p^G \underline{z}$$

is a simple $\mathrm{GL}(m, n)$-module, as desired. □

7 Lusztig's tensor product theorem

The purpose of this section is to establish the tensor product theorem for the quantum supergroup U_η analogous to that in [15]. Assume l is an odd number ≥ 3 and η

is a primitive lth root of unity. We draw most of the notation and standard results from [15].

Applying a similar argument as that for [15, Lemma 7.2], one obtains

Lemma 7.1. *Let $\mathfrak{V}_\eta$ be the subalgebra of U_η generated by $E_{\alpha_i}^{(l)}, F_{\alpha_i}^{(l)}, \begin{bmatrix} K_{\alpha_{m+n}}, c \\ l \end{bmatrix}$, $\begin{bmatrix} K_{\alpha_m}, c \\ l \end{bmatrix}$, $i \in [1, m+n) \setminus m$, $c \in \mathbb{Z}$. Assume $\underline{z} \in \mathbb{Z}^{m+n}$ with $\underline{z} = l\underline{z}_1$ for some $\underline{z}_1 \in \mathbb{Z}^{m+n}$. Let x be a maximal vector of $L_\eta(\underline{z})$. Then*

(a) $E_{\alpha_i}, F_{\alpha_i}, K_{\alpha_j} - 1$ act as 0 on $L_\eta(\underline{z})$ ($i \in [1, m+n)$, $j \in [1, m+n]$).

(b) Let $\nabla = \{y \in \mathcal{H}(L_\eta(\underline{z})) | E_{\alpha_i}^{(l)} y = 0 \quad \text{for all} \quad i \neq m, m+n\}$. Then $\nabla = \mathbb{C} \cdot x$.

(c) $L_\eta(\underline{z}) = \mathfrak{V}_\eta \cdot x$.

Let $\underline{z} \in \mathbb{Z}_+^{m+n}$. We can write uniquely $\underline{z} = \underline{z}_1 + l\underline{z}_2$, where $\underline{z}_1 \in \mathbb{Z}_l^{m+n}$ and $\underline{z}_2 \in \mathbb{Z}_+^{m+n}$. Then applying a similar proof as that of [15, 7.4], we get

Theorem 7.2. *The U_η-modules $L_\eta(\underline{z})$ and $L_\eta(\underline{z}_1) \otimes L_\eta(l\underline{z}_2)$ are isomorphic.*

Proposition 7.3. *For any $\underline{z} = (z_1, \cdots, z_{m+n}) \in \mathbb{Z}_+^{m+n}$, $L_\eta(l\underline{z})$ is a simple $\overline{U}(\mathfrak{g}_{\bar{0}})_\mathbb{C}$-module with highest weight $\underline{z}$.*

Proof. Let I be the two-sided ideal of U_η generated by E_{α_i}, F_{α_i}, $K_{\alpha_j} - 1$, $i \in [1, m+n), j \in [1, m+n]$. In view of the proof of [15, 7.5], we have a unique superalgebra epimorphism $\phi : \overline{U}(\mathfrak{g}_{\bar{0}})_\mathbb{C} \longrightarrow U_\eta/I$ such that

$$\begin{aligned} \phi(e_{\alpha_i}) &= \overline{E_{\alpha_i}^{(l)}}, \\ \phi(f_{\alpha_i}) &= \overline{F_{\alpha_i}^{(l)}}, \end{aligned} \quad i \in [1, m+n) \setminus m,$$

$$\phi(h_{\alpha_i}) = \overline{\begin{bmatrix} K_{\alpha_i}; 0 \\ l \end{bmatrix}}, i \in [1, m+n].$$

Since $\overline{E_{\alpha_i}^{(l)}}$, $\overline{F_{\alpha_i}^{(l)}}$, $\overline{\begin{bmatrix} K_{\alpha_m}; 0 \\ l \end{bmatrix}}$, $\overline{\begin{bmatrix} K_{\alpha_{m+n}}; 0 \\ l \end{bmatrix}}$ generate U_η/I as an algebra, ϕ is surjective. By Lemma 7.1, $L_\eta(l\underline{z})$ is a simple U_η/I-module and hence the pull back by ϕ is a simple $\overline{U}(\mathfrak{g}_{\bar{0}})_\mathbb{C}$-module. Now let x be a maximal vector of $L_\eta(l\underline{z})$. Then by [15, 3.2(a)], we have $h_{\alpha_i} \cdot x = z_i x$, $i \in [1, m+n]$. $\square$

Recall the notation $U(\mathfrak{g}_{\bar{0}})_\mathbb{Z}$ and $U_\mathcal{B}$ in Sec. 6.2. Following Lusztig [18], we define the Frobenius morphism for the quantum supergroup $U_\mathcal{B}$.

Theorem 7.4. *(compare [18, Th. 8.10]) There is a unique $\mathcal{B}$-superalgebra homomorphism $\chi : U_\mathcal{B} \longrightarrow U(\mathfrak{g}_{\bar{0}})_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{B}$ such that $\chi(E_{\alpha_i}^{(N)})$ is $E_{\alpha_i}^{(N/l)}$ if l divides N and is zero otherwise; $\chi(F_{\alpha_i}^{(N)})$ is $F_{\alpha_i}^{(N/l)}$ if l divides N and is zero otherwise; $\chi(\begin{bmatrix} K_{\alpha_i}; 0 \\ N \end{bmatrix})$ is $\begin{bmatrix} K_{\alpha_i}; 0 \\ N/l \end{bmatrix}$ if l divides N and is zero otherwise ($i \in [1, m+n]$); $\chi(K_{\alpha_i}^{\pm 1}) = K_{\alpha_i}^{\pm 1} (i \in [1, m+n])$.*

Proof. Note that $U(\mathfrak{g}_{\bar{0}})_{\mathbb{Z}}$ is generated by the even elements

$$E_{ij}^{(N)}, F_{ij}^{(N)}, \begin{bmatrix} K_{\alpha_m}; 0 \\ t \end{bmatrix}, \begin{bmatrix} K_{\alpha_{m+n}}; 0 \\ t \end{bmatrix}, K_{\alpha_s}^{\pm 1}, (i,j) \in \mathcal{I}_0, t, N \geq 0, s \in [1, m+n]$$

with the same relations as those for U given in Sec.5, with v replaced by 1. Then it suffices to check that χ preserves the relations Lemma 4.11 (1)-(5), Sec.5 (f1)-(f6), (g1)-(g5), (h1)-(h6). Note that χ carries odd elements to zero and $v^l = 1$ in $\mathcal{B}$. Then it is clear that each relation involving only even generators is preserved by χ, so that χ is a $\mathcal{B}$-superalgebra homomorphism. Clearly χ is surjective. $\square$

The arguments leading to [14, 6.7(a)] can be repeated to show that there is a $\mathbb{C}$-algebra epimorphism from $U(\mathfrak{g}_{\bar{0}})_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{B} \otimes_{\mathcal{B}} \mathbb{C}$ into $\overline{U}(\mathfrak{g}_{\bar{0}})_{\mathbb{C}}$ sending K_{α_i} to 1. Composing this epimorphism with $\chi \otimes 1_{\mathbb{C}}$, we see that $L_\eta(l\underline{z})$ becomes a simple U_η-module of highest weight $\underline{z}$.

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